Mechanics of Earthquakes and Faulting

Lectures 2 & 3, 2021

www.geosc.psu.edu/Courses/Geosc508

- Discussion of Handin, JGR, 1969 and Chapter 1 Scholz, 2019.
- Stress analysis and Mohr Circles
- Coulomb Failure Criteria for Fracture and Friction
- Brittle Failure, Stress-Strain curves for some common material behaviors
- Theoretical strength of materials
- Defects
- Stress concentrations
- Griffith failure criteria

- Energy balance for crack propagation
- Stress intensity factors
- Static & dynamic fracture mechanics
- Critical energy release rate for rupture
- Process zone
- Crack models

- Discussion of Handin, JGR, 1969 and Chapter 1 Scholz

- Rankine's condition: what is it?
- What is the coefficient of internal friction?
- How are material properties a function of the state of stress? What did Handin mean by this statement on p. 5344 (top left)?
- Rocks pass from brittle to ductile deformation mechanism with what changes in strain rate, temperature and pressure?   What does this mean for the linearity of the Mohr envelope?

Imagine that you're in a restaurant with some friends. The owner stops by to say hello and after hearing that you're a geophysicist she challenges you to write down the Shear and Normal Stress on a Plane of Arbitrary Orientation given the principal stresses.

She calls the waiter over and he gives you a couple extra napkins and a pencil and says, don't worry about the third dimension because that's always in the fault plane for simple (Andersonian) faulting. So you know that you can just use two principal stresses. The maximum and minimum stress. Go ahead and call them σ_1 and σ_2

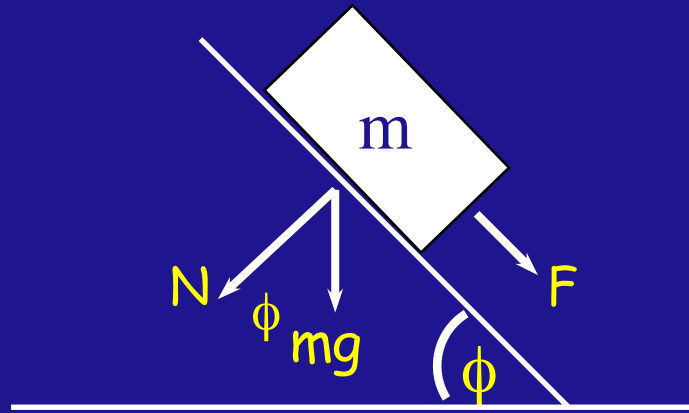**Ok, get to work! You've got to finish before he brings the drinks**

Sketch in plane P

$$\sigma(2\alpha) =$$

$$\tau(2\alpha) =$$
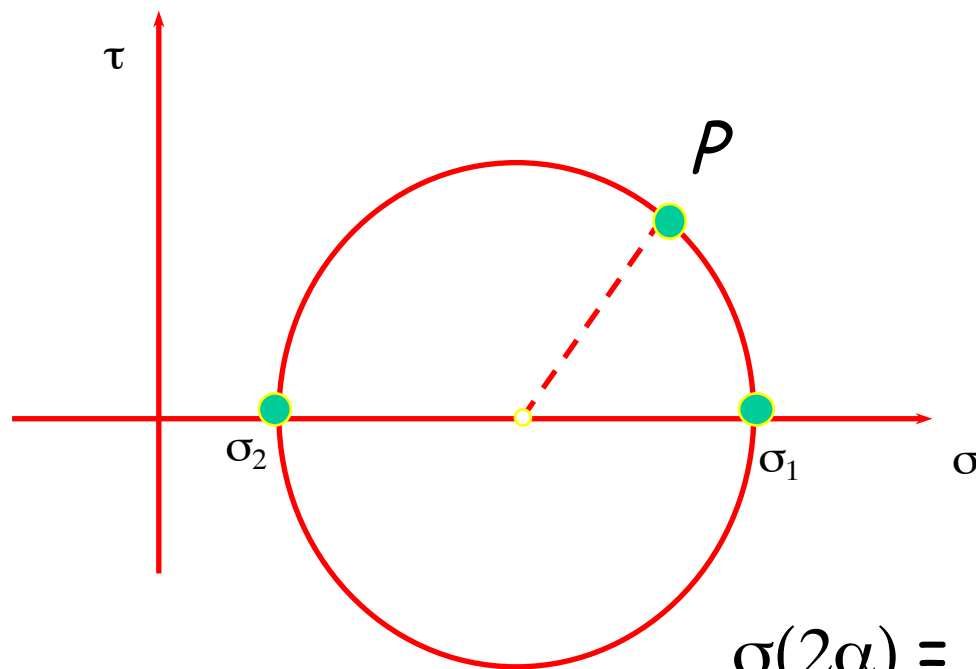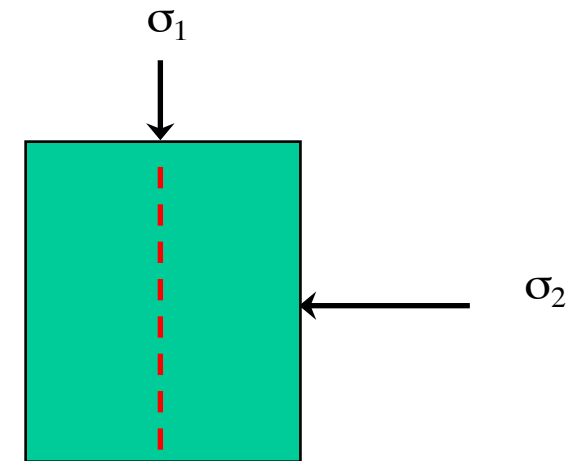
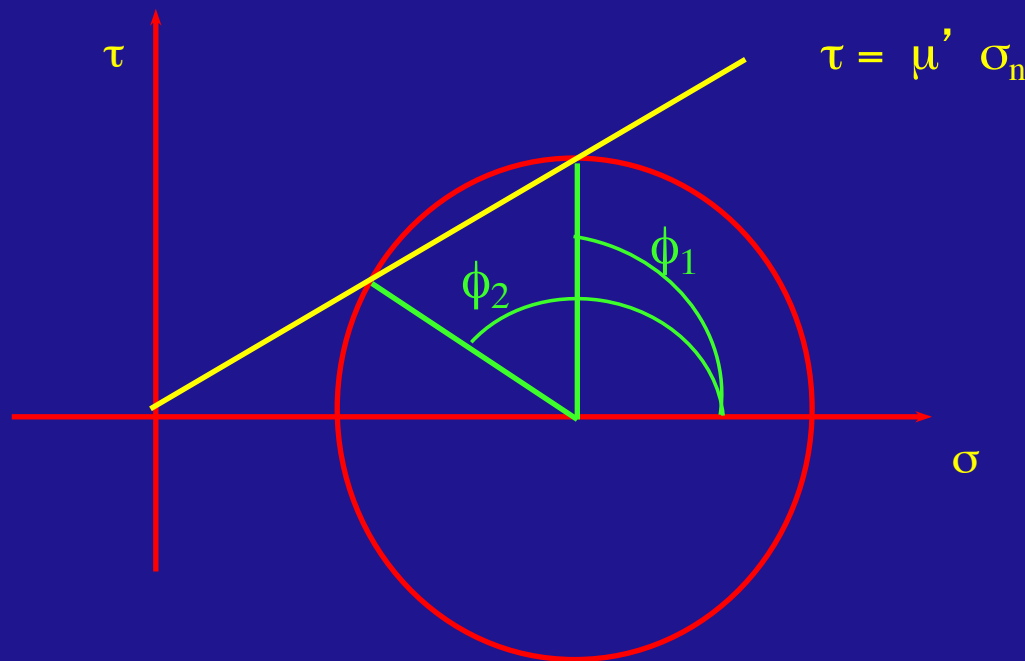Mass on an inclined plane, as in a simple friction experiment.

Coefficient of friction μ = F/N

Amonton's Law:
$$F/N = \mu = \tan \phi$$

In terms of stresses: $\tau = \mu \, \sigma_n$

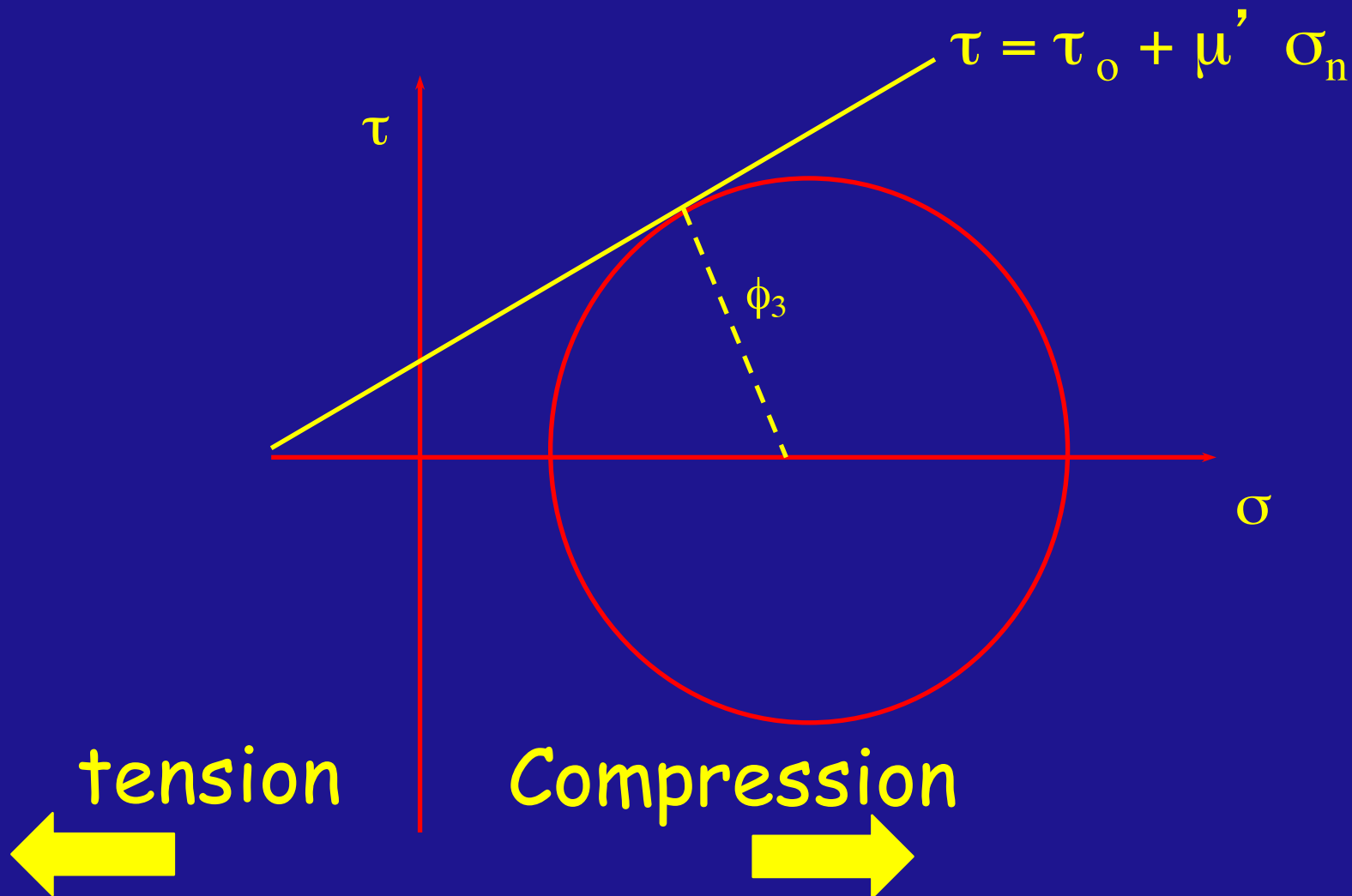This failure criterion can be plotted on a Mohr diagram

$\tau = \mu' \, \sigma_n$

For this stress state, the Mohr plot shows that frictional failure would occur on any plane of orientation between ϕ_1 and ϕ_2

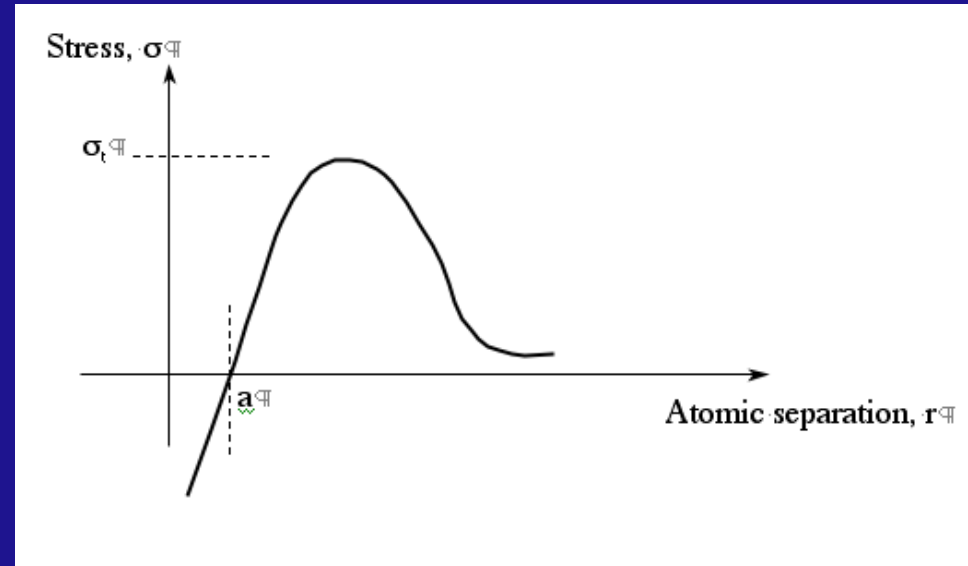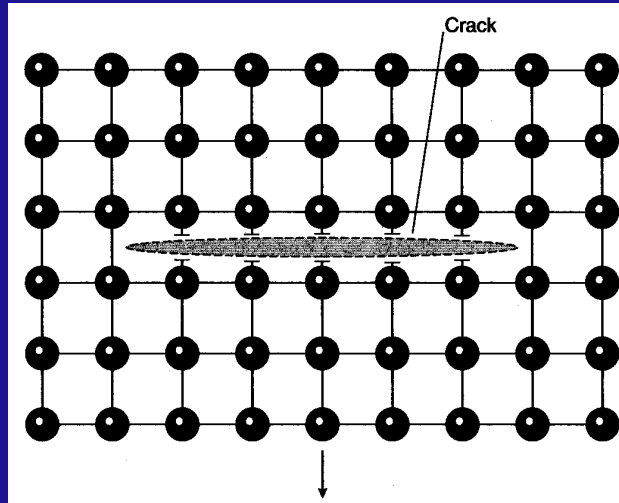The Coulomb criterion can be used to calculate uniaxial breaking strength C_o

$$C_o = 2\,\tau_o\,[\,(\mu^2 + 1)^{1/2} + \mu\,]$$

And things get more complicated if you get into the tensile stress field

$$\tau = \tau_o + \mu'\,\sigma_n$$

τ

ϕ_3

σ

tension | Compression

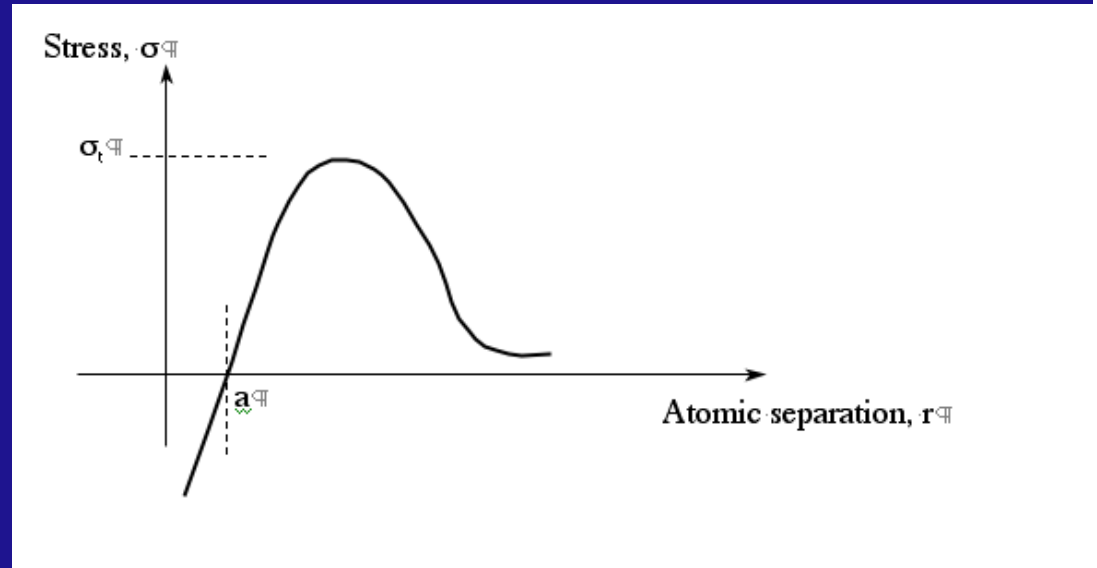# Theoretical strength of materials

- Defects
- Stress concentrations
- Griffith failure criteria
- Energy balance for crack propagation
- Stress intensity factor

Theoretical strength of materials

- Defects
- Stress concentrations
- Griffith failure criteria
- Energy balance for crack propagation
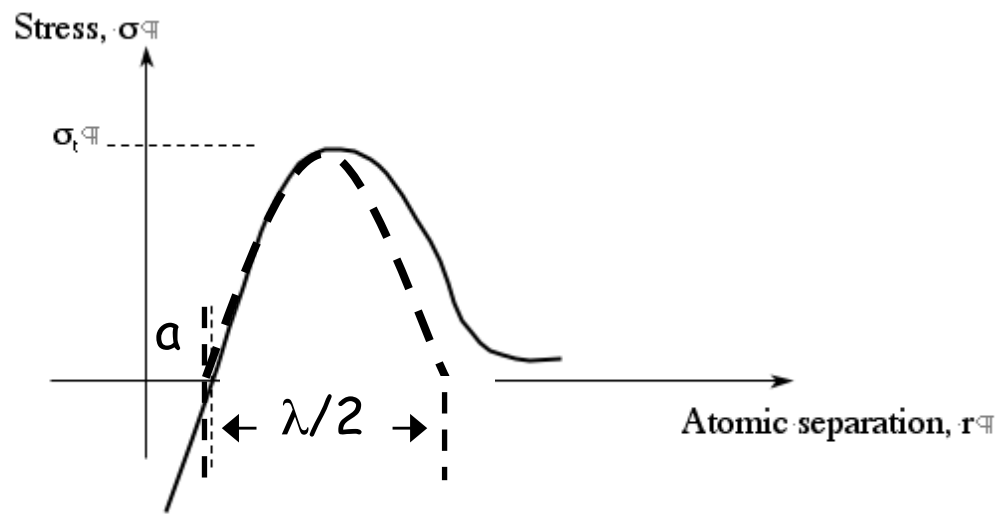- Stress intensity factor

Start by thinking about the theoretical strength of materials –and take crystals as a start. The strength of rocks and other polycrystalline materials will also depend on cementation strength and grain geometry so these will be more complex.

$$\sigma = \sigma_t \sin\left(\frac{2\pi(r-a)}{\lambda}\right),$$

$$\frac{d\sigma}{d(r-a)} = \frac{E}{a} = \frac{2\pi}{\lambda}\sigma_t \cos\left(\frac{2\pi(r-a)}{\lambda}\right),$$

Since $\dfrac{(r-a)}{\lambda} \ll 1$, we can re-write this as: $\sigma_t = \dfrac{E\lambda}{2\pi a}$

Theoretical strength, σ_t, of simple crystals:
Bonds must break along a lattice plane

Consider a tensional stress field, and take *a* as the equilibrium lattice spacing.
Approximate the region around the peak strength as a sinusoid, wavelength λ
Then, for small changes in lattice spacing: the rate of stress change is related to E.

$$\sigma = \sigma_t \sin\left(\frac{2\pi(r-a)}{\lambda}\right),$$

$$\frac{d\sigma}{d(r-a)} = \frac{E}{a} = \frac{2\pi}{\lambda}\sigma_t \cos\left(\frac{2\pi(r-a)}{\lambda}\right),$$

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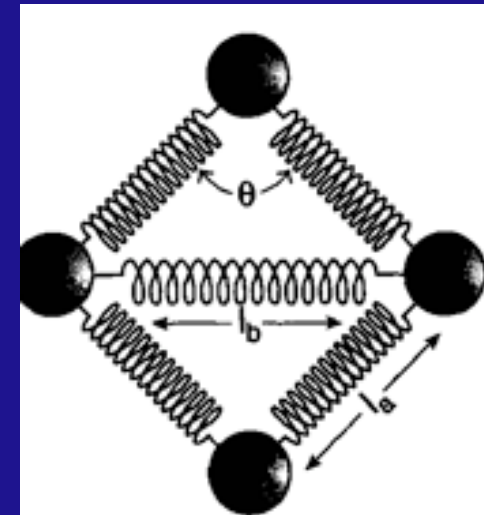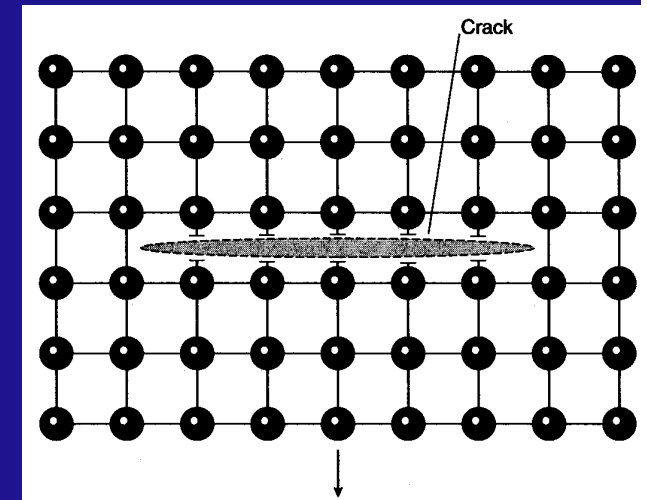
Tensile Strength of single x'l, by our approximation:

$$\sigma_t = \frac{E\lambda}{2\pi a}$$

The strain energy and stress is zero at thermodynamic equilibrium, which occurs at r= 3a/2 and since a ≈ λ, the theoretical strength is about E/2π. (See Scholz, Ch. 1.1 for additional details).

$$\sigma_t \approx \frac{E}{2\pi}$$

- This type of calculation was carried out in the early 1900's and people immediately realized that there was a problem.
- Experiments showed that E was on the order of 10's of GPa, whereas the tensile strength of most materials is closer to 10's of MPa.

Crack

- Griffith proposed a solution in two classic papers in the early 1920's –but the proof of his ideas had to wait until the invention of the electron microscope.

Bottom line: Defects.

Defects severely reduce the strength of brittle materials relative to the theoretical estimate. Flaws exist at all scales from atomic to the specimen size (laboratory sample size or continent scale, in the case of plate tectonics)

Stress concentrations around defects cause the local stress to reach the theoretical strength.

Two types of defects cause two types of deformation:
- cracks and crack propagation lead to brittle deformation;
- dislocations and other types of atomic misregistration lead to plastic flow and 'ductile' deformation.

Brittle deformation generally leads to catastrophic failure and separation of lattice elements.

Plastic flow produces permanent deformation without loss of lattice integrity.

Scholz generalizes these modes of deformation to make a connection with lithospheric deformation.

The upper lithosphere deforms primarily by brittle mechanisms and can be referred to as the **schizosphere** (lit. the broken part), whereas

the lower lithosphere deforms by ductile mechanisms and can be classified as the **plastosphere**.

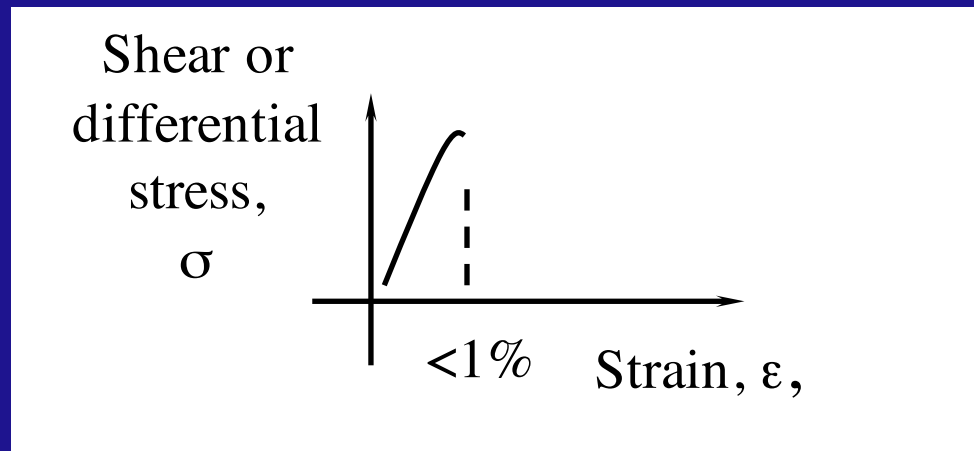**Rheology and Deformation.   Definitions.**

The terms brittle and ductile can be defined in a number of ways. One def. is given above.   Another important operational definition involves the stress-strain characteristics and the dependence of strength on mean (or normal stress).
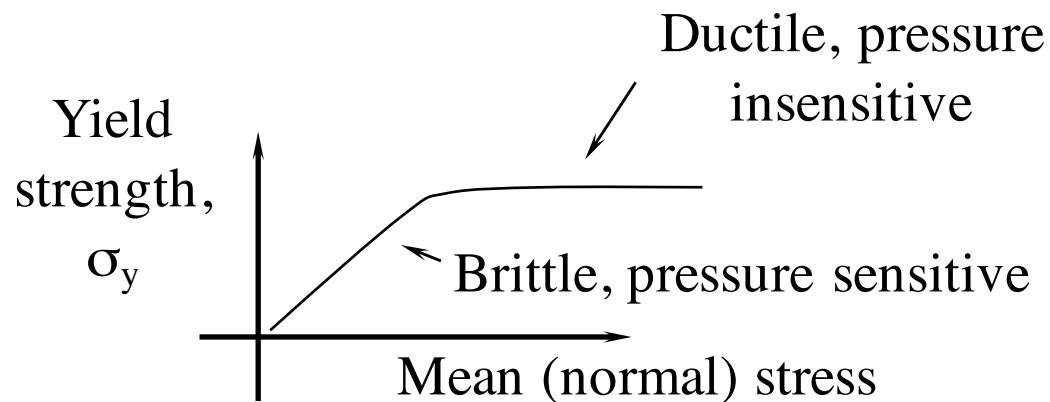
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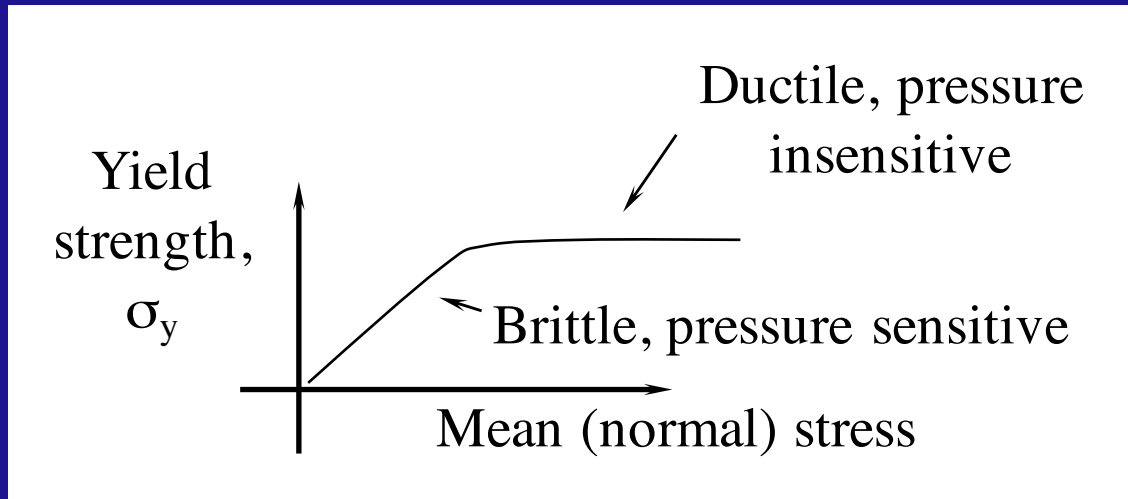
Rheology and Deformation.   Definitions.

The term 'brittle' is also used to describe materials that break after very little strain.

Shear or differential stress, σ

<1%   Strain, ε,

Fracture toughness describes a material's ability to deform without breaking.
•Brittle materials (like glass or ceramics) have low toughness.
•Plastics have high toughness

What causes the pressure sensitivity of brittle deformation?

- Volume change. Brittle deformation involves volume change –dilatancy or compaction.

- 'Dilation' means volume increase. Dilatancy describes a shear induced volume increase. The term was introduced to describe deformation of granular materials – but dilation also occurs in solid brittle materials via the propagation of cracks.

- Work is done to increase volume against the mean stress during brittle deformation, thus the pressure sensitivity of brittle deformation.

- Ductile deformation occurs without macroscopic volume change, due to the action of dislocations. Dislocation motion allows strain accommodation.
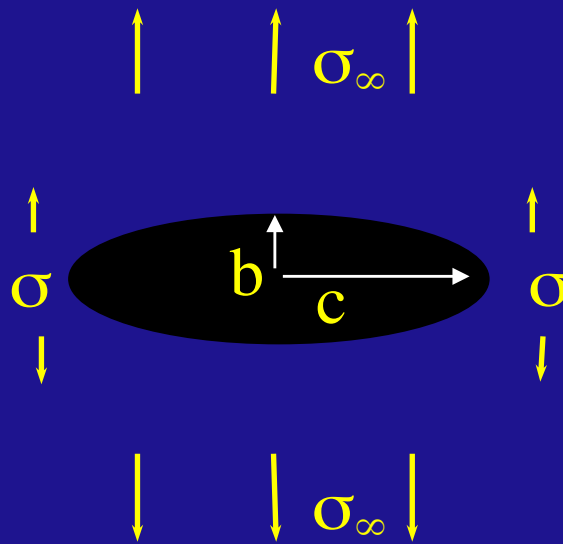
Stress concentrations around defects.

In general, the stress field around cracks and other defects is quite complex, but there are solutions for many special cases and simple geometries

Scholz gives a partial solution for an elliptical hole in a plate subject to remote uniform tensile loading (ρ is the local curvature)

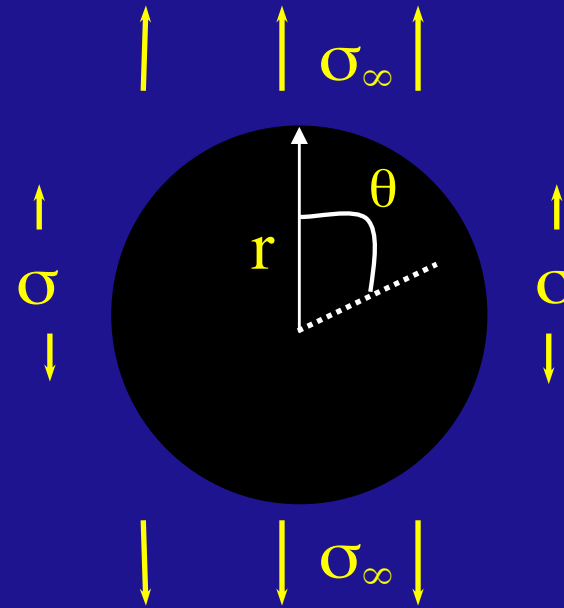$$\sigma = \sigma_\infty \left(1 + 2\frac{c}{b} \right)$$

$$\sigma = 2\sigma_\infty \left(1 + 2\sqrt{\frac{c}{\rho}} \right)$$

Crack tip stresses

σ_∞

σ σ

b
c

σ_∞

Malvern (1969) gives a full solution for a circular hole or radius r = a

$$\sigma_{\theta\,Max} = \sigma_\infty\left(a,\ \frac{\pi}{2}\right) = 3\sigma_\infty$$

Full solution for a circular hole of radius r=a

$$\sigma_r = \frac{\sigma_\infty}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_\infty}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{\sigma_\infty}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_\infty}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$
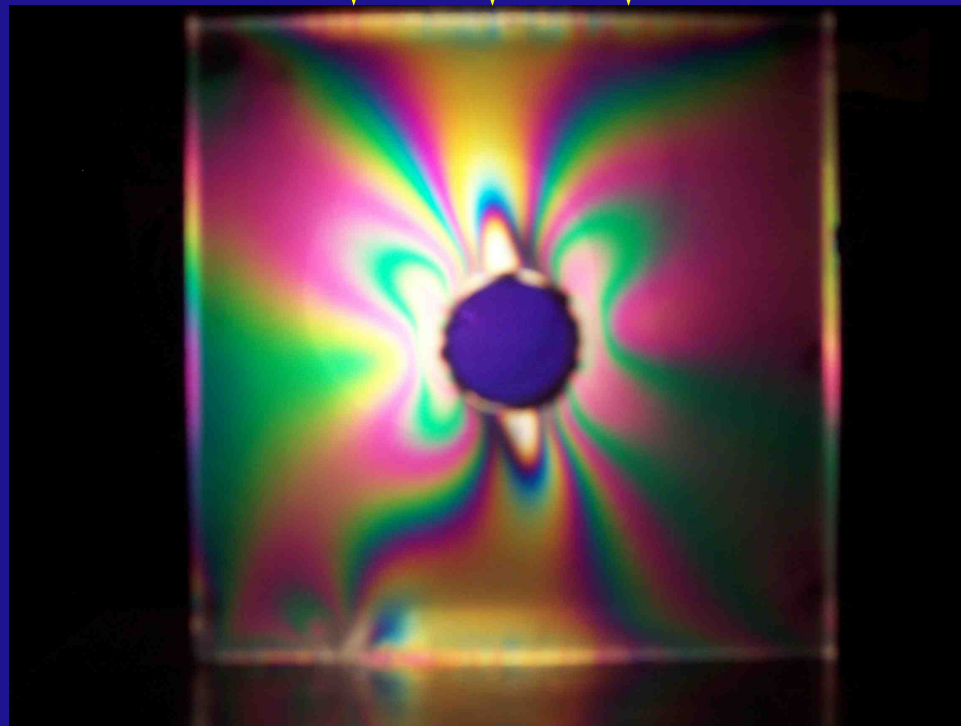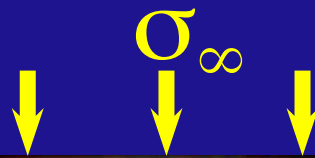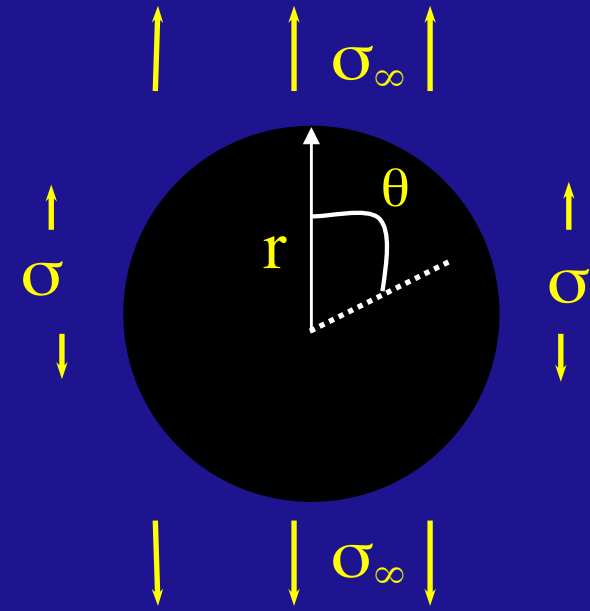
$$\tau_{r\theta} = -\frac{\sigma_\infty}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

Full solution for a circular hole of radius r=a

$$\sigma_r = \frac{\sigma_\infty}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_\infty}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{\sigma_\infty}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_\infty}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{\sigma_\infty}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

Bond separation and specific surface energy.

- Fracture involves creation of new surface area.
- The specific surface energy is the energy per unit area required to break bonds.

Two surfaces are created by separating the material by a distance $\lambda/2$ and the work per area is given by stress times displacement.

$$2\gamma = \int_0^{\frac{\lambda}{2}} \sigma_t \sin\left(\frac{2\pi(r-a)}{\lambda}\right) \ d(r-a)$$

$$= \frac{\lambda \sigma_t}{\pi}$$

This yields the estimate: .

$$\gamma = \frac{Ea}{4\pi^2}$$

The surface energy is a fundamental physical quantity and we will return to it when we talk about the energy balance for crack propagation and the comparison of laboratory and seismic estimates of G, the fracture energy.

Can crack mechanics help to solve, quantitatively, the huge discrepancy between the theoretical (~10 GPa) and observed (~10 MPa) values of tensile strength?

For a far field applied stress of σ_∞, we have crack tip stresses of

$$\sigma \approx 2\sigma_\infty \sqrt{\frac{c}{\rho}}$$

Taking σ as σ_t, we can combine the relations for

theoretical strength

$$\sigma_t = \frac{E\lambda}{2\pi a}$$

and surface energy

$$\gamma = \frac{Ea}{4\pi^2}$$
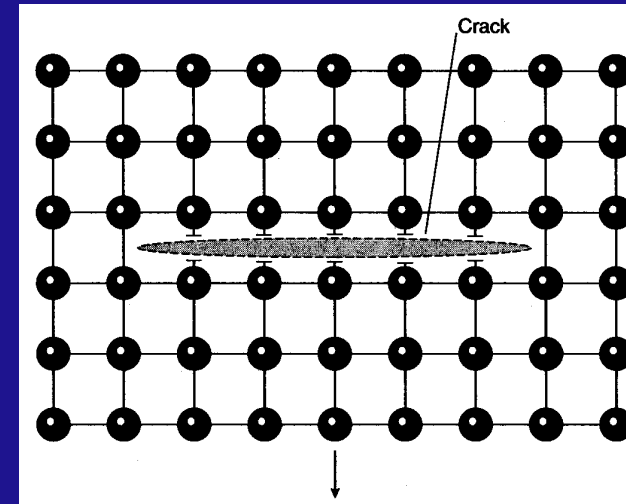
to get:

$$\sigma_t = \sqrt{\frac{E\gamma}{a}}$$

If we take crack radius as approx. equal to *a, the lattice dimension*, then setting σ_t, equal to σ at the crack tip, we have:

$$2\sigma_\infty \sqrt{\frac{c}{a}} = \sqrt{\frac{E\gamma}{a}}, \ which \ yields: \ \sigma_\infty = \sqrt{\frac{E\gamma}{4c}}$$

Taking σ_∞ of 10 MPa, E= 10 GPa and γ of 4×10^{-2} J/m^2, gives a crack half length c of 1 micron.

•Griffith proposed that all materials contain preexisting microcracks, and that stress will concentrate at the tips of the microcracks

•The cracks with the largest elliptical ratios will have the highest stress, and this may be locally sufficient to cause bonds to rupture
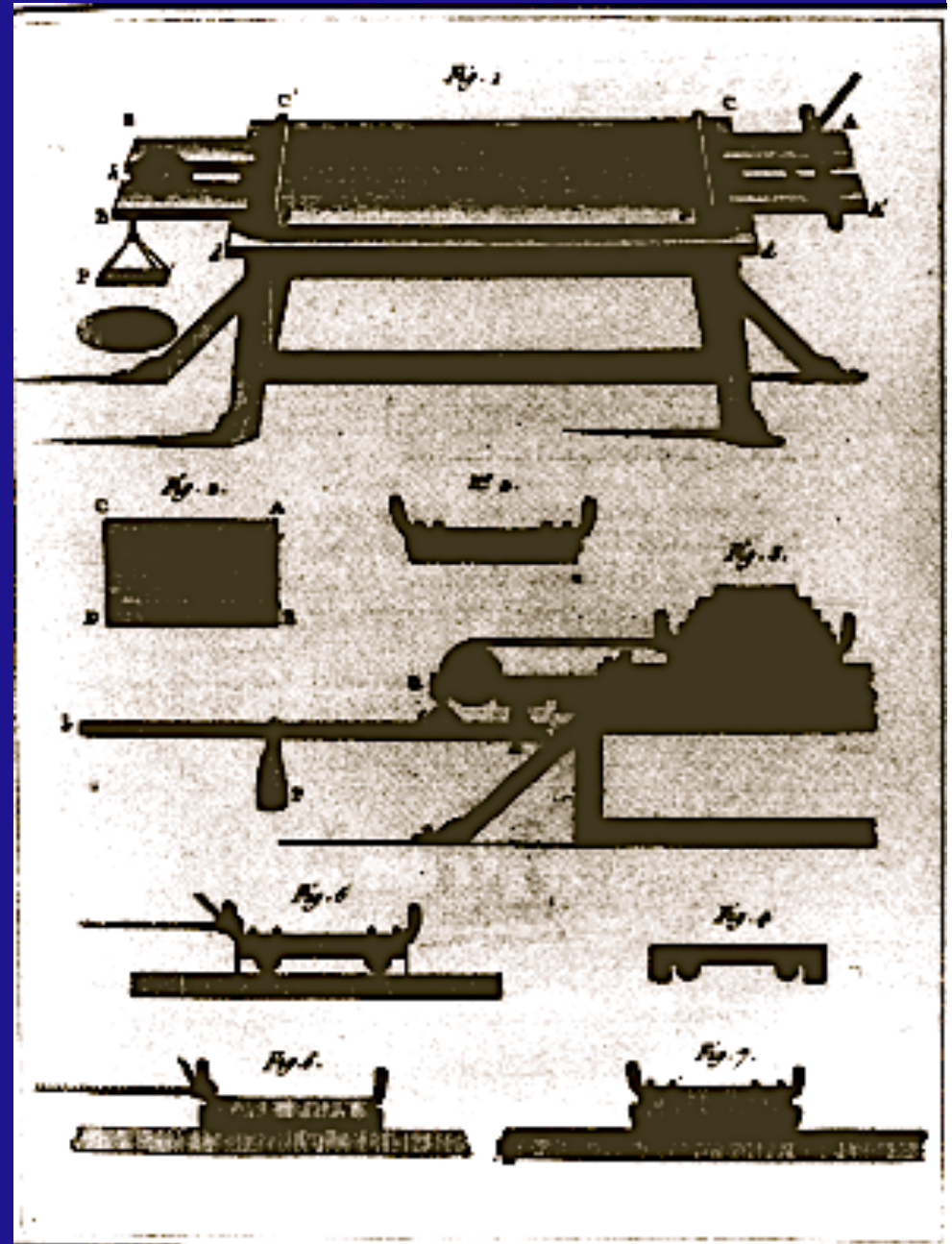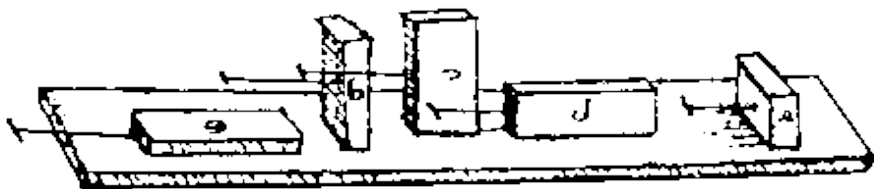

Crack

- As the bonds break, the ellipticity increases, and so does the stress concentration
- The microcrack begins to propagate, and becomes a real crack
- Today, microcracks and other flaws, such as pores or grain boundary defects, are known as Griffith defects in his honor
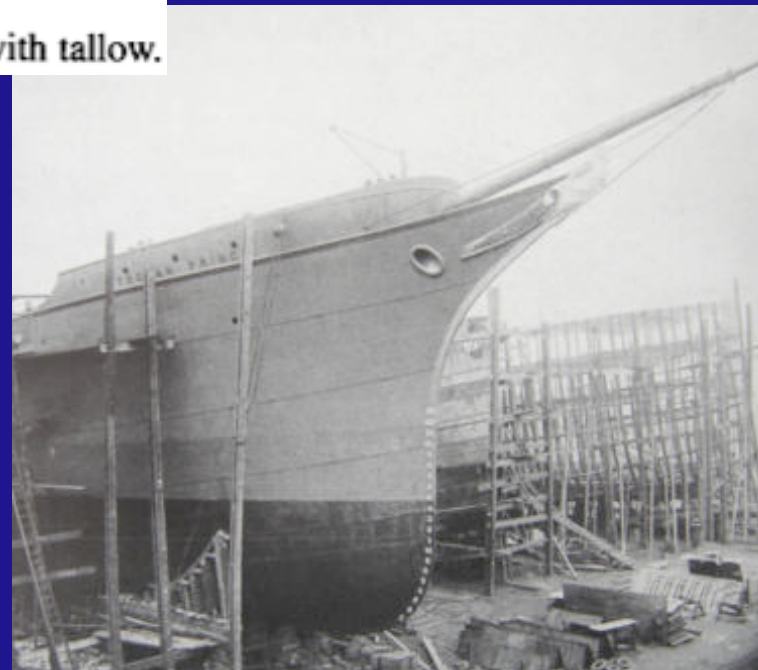

Crack-Tip Stress Zone
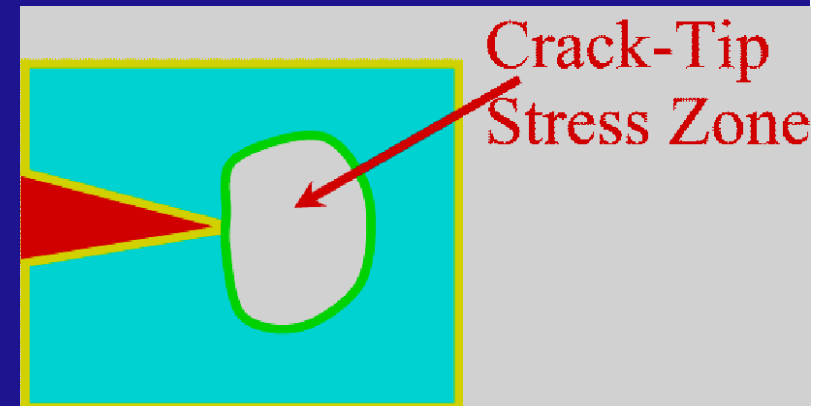
C. A. Coulomb (1736-1806)

Brittle Failure and Time dependence of "static" friction

Table 9.1

	T (time of repose, min)	$A+mT^a$ (static friction force, lbf)
I$^{\text{re}}$ observation	0	A=502
II$^{\text{e}}$	2	790
III$^{\text{e}}$	4	866
IV$^{\text{e}}$	9	925
V$^{\text{e}}$	26	1,036
VI$^{\text{e}}$	60	1,186
VII$^{\text{e}}$	960	1,535

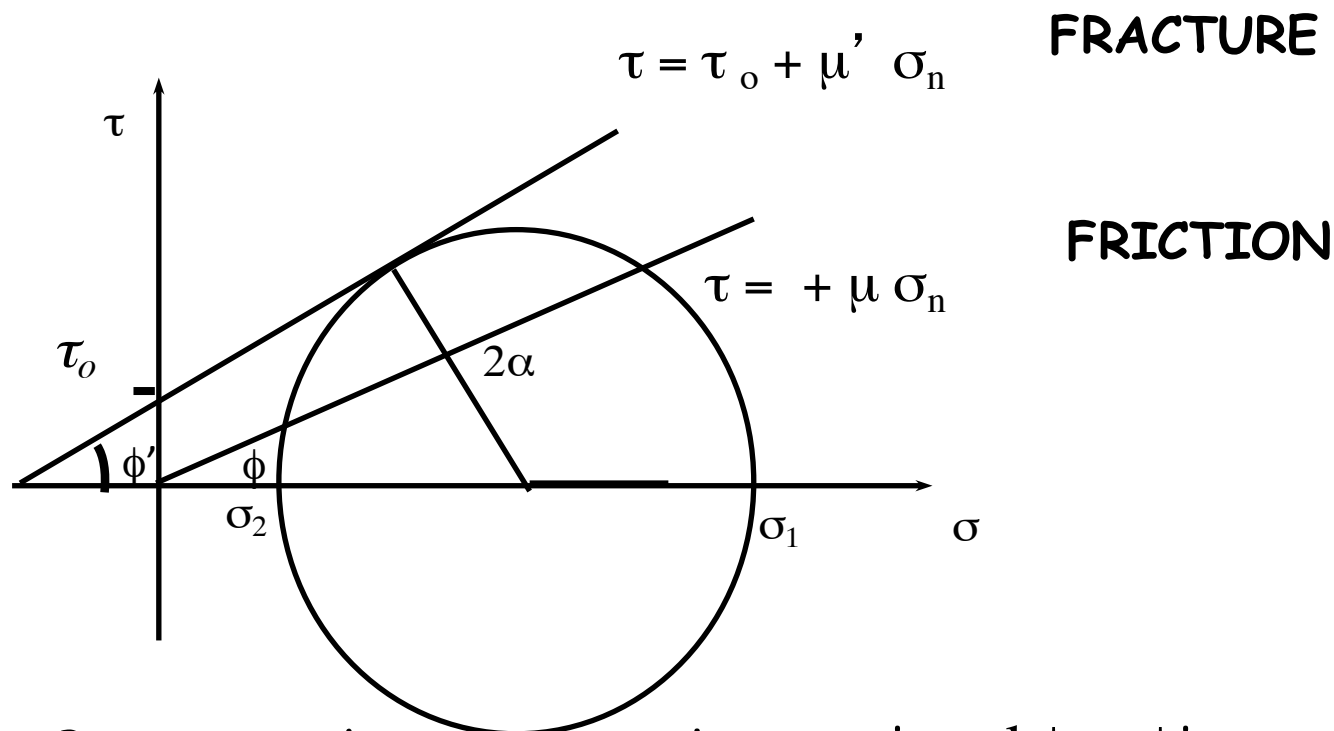static friction of two pieces of well-worn oak lubricated with tallow.

Shear and Normal Stress on a Plane of Arbitrary Orientation --written in terms of **Principal Stresses**:

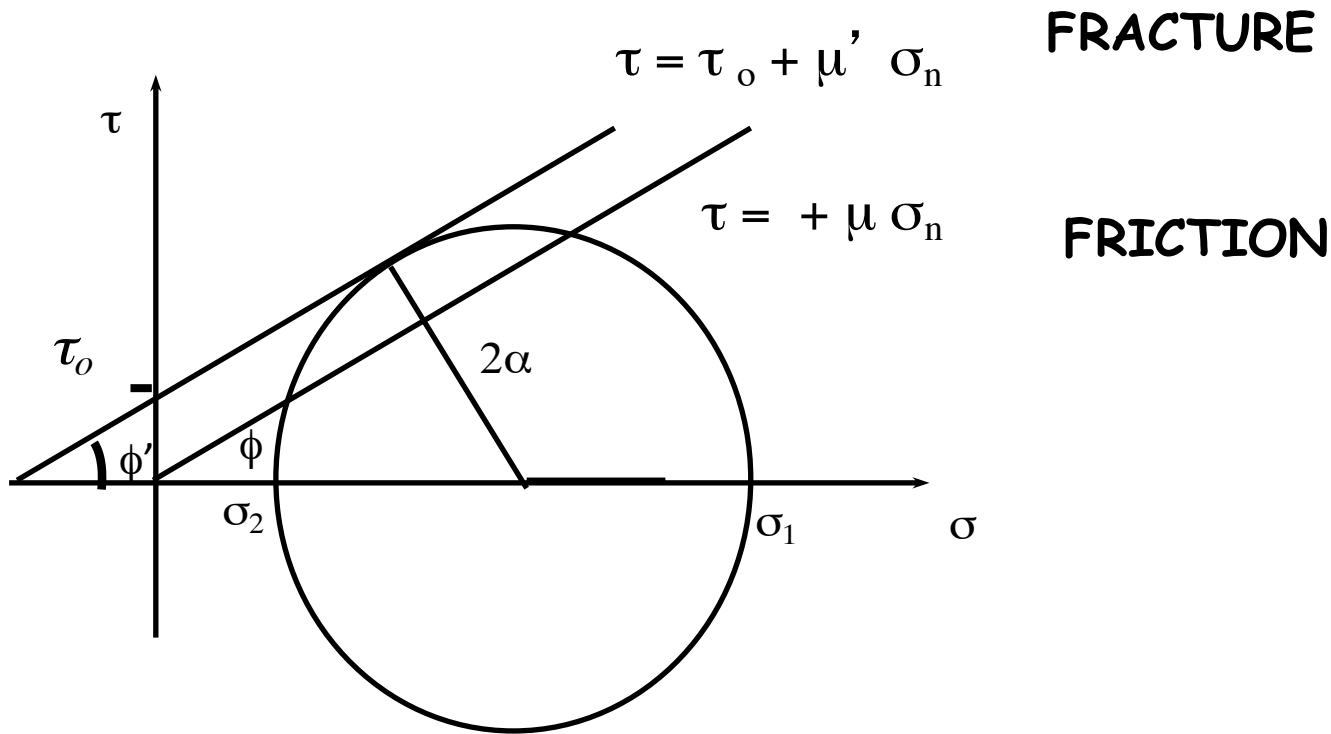$$\sigma = \frac{(\sigma_1 + \sigma_2)}{2} + \frac{(\sigma_1 - \sigma_2)}{2} \cos 2\alpha$$

$$\tau = \frac{\sigma_1 - \sigma_2}{2} sin(2\alpha)$$ **Mohr Circle.**

FRACTURE

$\tau = \tau_o + \mu' \, \sigma_n$
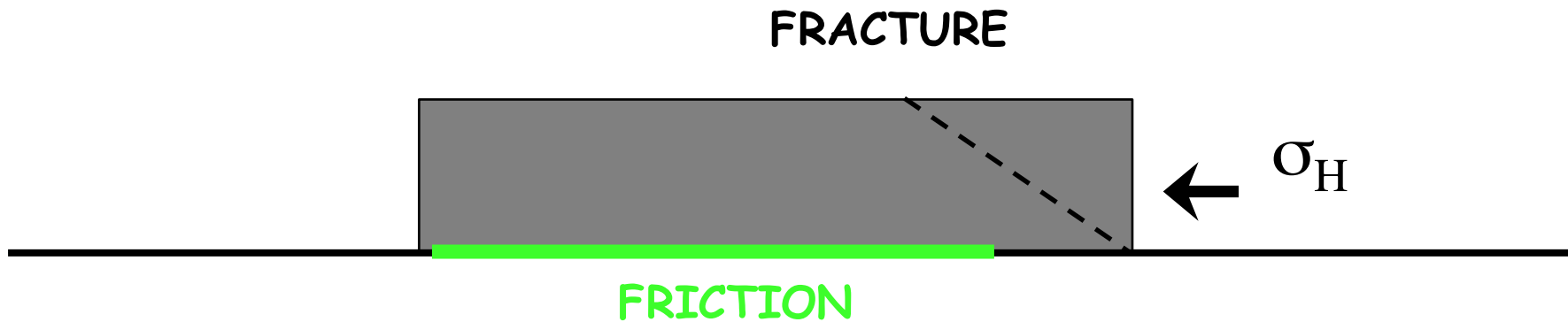
FRICTION

$\tau = \, + \mu \, \sigma_n$

But, note that in general: $\mu \neq \mu'$ and $\phi \neq \phi'$
That is, the coefficient of sliding friction is not necessarily
equal to the coefficient of internal friction.

$$\tau = \tau_o + \mu' \sigma_n \qquad \text{FRACTURE}$$

$$\tau = + \mu \sigma_n \qquad \text{FRICTION}$$

2α

τ

τ_o

ϕ' ϕ

σ_2 σ_1 σ

What is the longest block that can be slid in frictional contact?
Fracture will occur first for very long blocks

FRACTURE

FRICTION

σ_H

What is the longest block that can be slid in frictional contact?
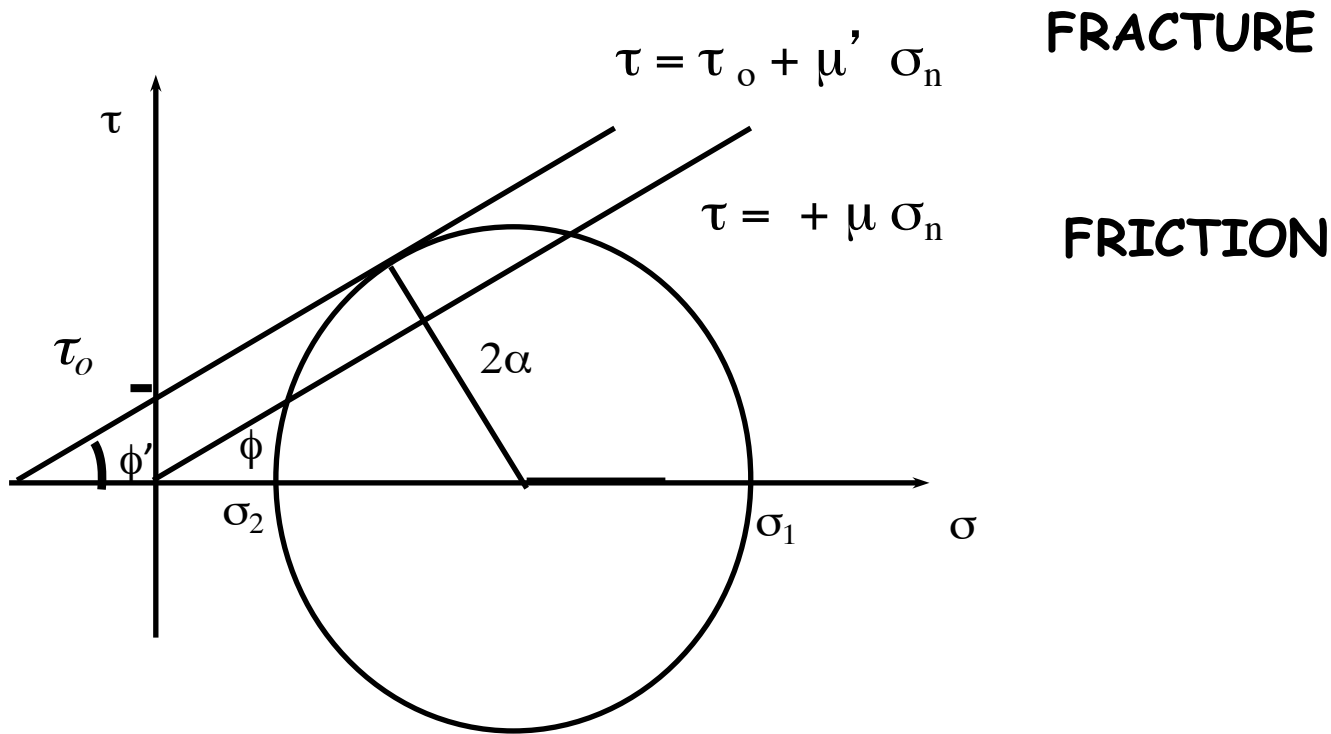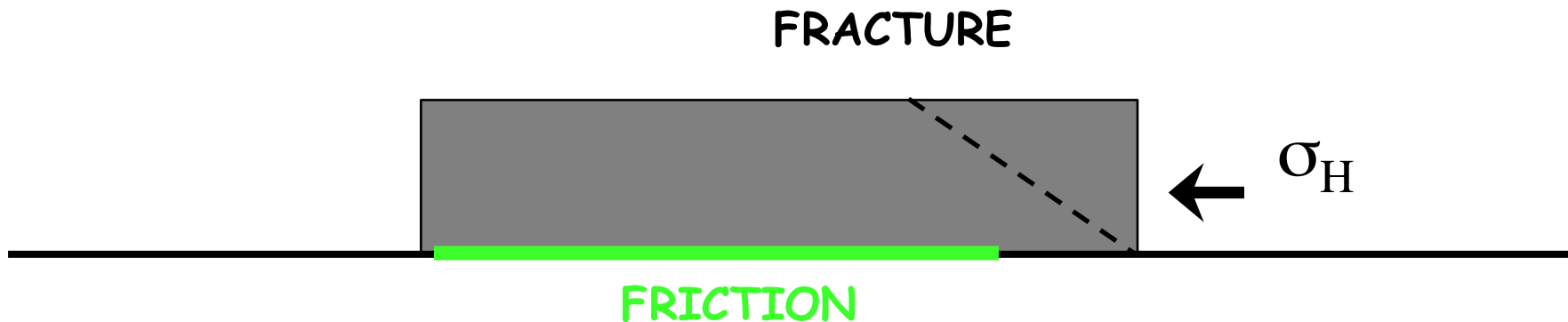Fracture will occur first for very long blocks

FRACTURE

σ_H

FRICTION

What is the longest block that can be slid in frictional contact?
Fracture will occur first for very long blocks

FRACTURE

σ_H

FRICTION

$$\tau = \tau_o + \mu' \, \sigma_n$$

FRACTURE

$$\tau = + \mu \, \sigma_n$$

FRICTION

τ

τ_o

2α

ϕ' ϕ

σ_2

σ_1

σ

What is the longest block that can be slid in frictional contact?
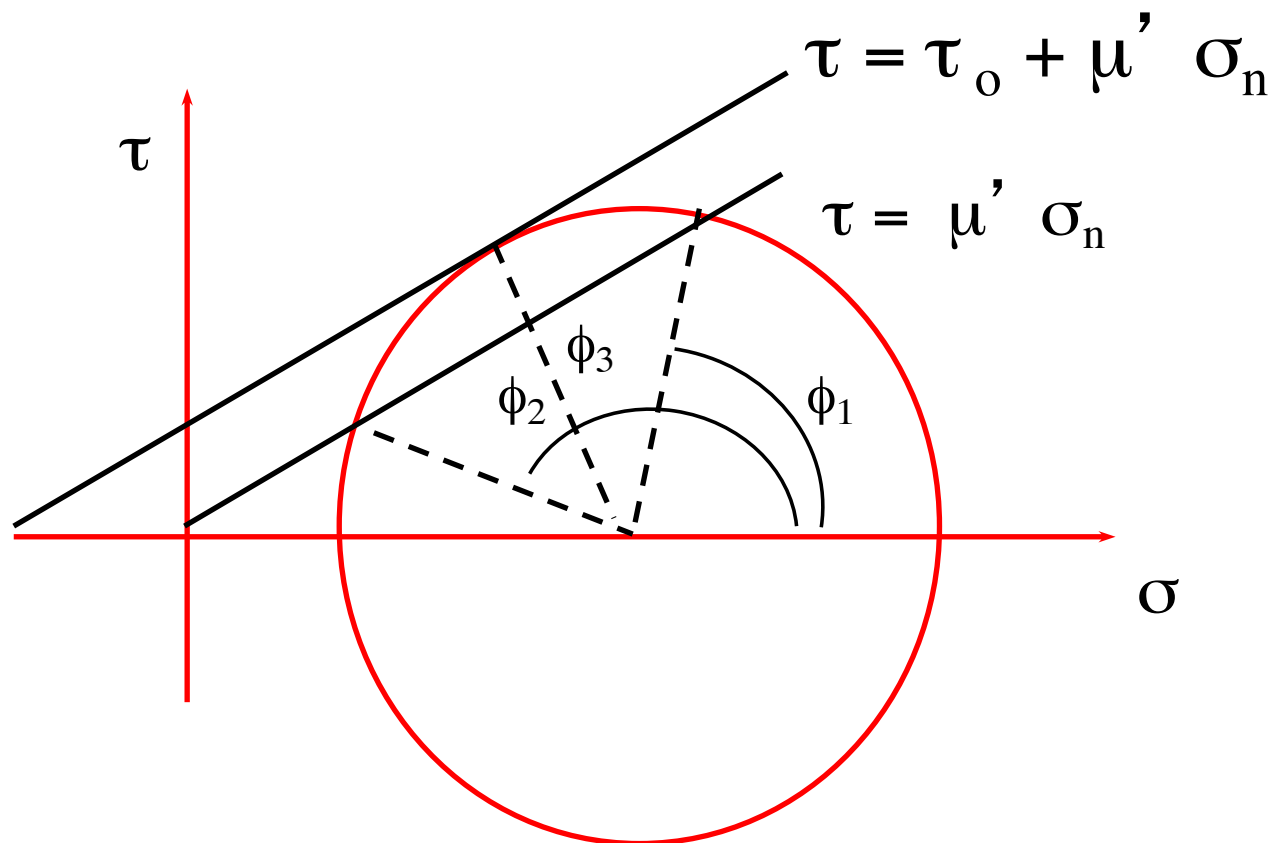Fracture will occur first for very long blocks

FRACTURE

σ_H
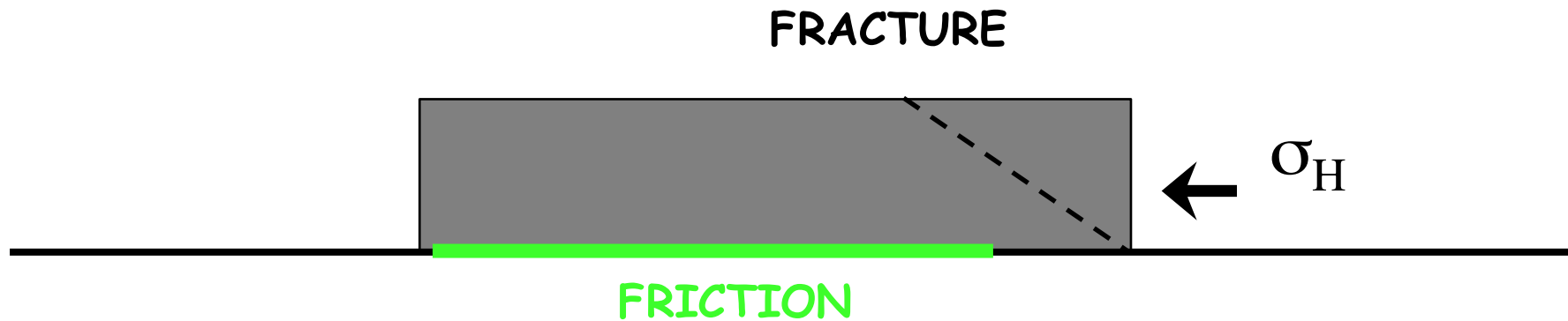
FRICTION

The Coulomb and Frictional failure criteria may be
considered together, on a Mohr diagram

This shows that pre-existing planes of weakness, of
orientations from ϕ_1 to ϕ_2, will fail by frictional slip prior
to a new fracture forming at orientation ϕ_3.

$$\tau = \tau_o + \mu' \sigma_n$$
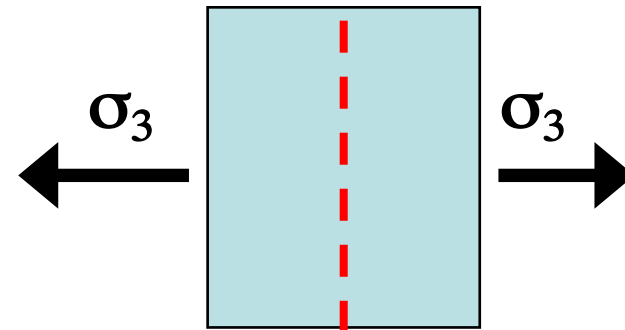
$$\tau = \mu' \sigma_n$$

Macroscopic Failure Criteria: Faulting, Fracture, Friction

These are not in general rheologic laws, but rather relationships between principal stresses (or applied stresses) at failure.

In tension, we have failure at $\sigma_3 = -T_o$, where T_o is the tensile strength.

How does tensile strength depend on pressure, applied stresses, temperature?


What about under compression?
- Pressure-dependent brittle failure
- Failure stress is higher under higher normal stress.

Handin, J. (1969), *On the Coulomb-Mohr failure criterion*, JGR. *74*, 5343-5348

Griffith, Modified Griffith: to explain curvature and transitional (hydrid) fractures. Based on stress concentrations at crack tips.
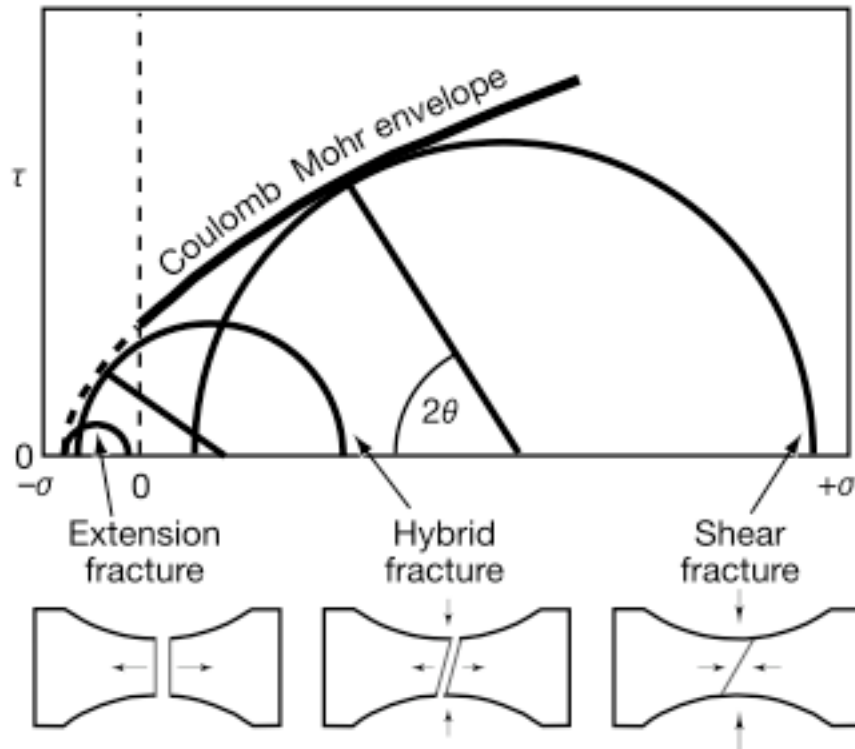

Ramsey & Chester: Hybrid fracture and the transition from extension fracture to shear fracture, *Nature* **428**, 63-66 (4 March 2004)
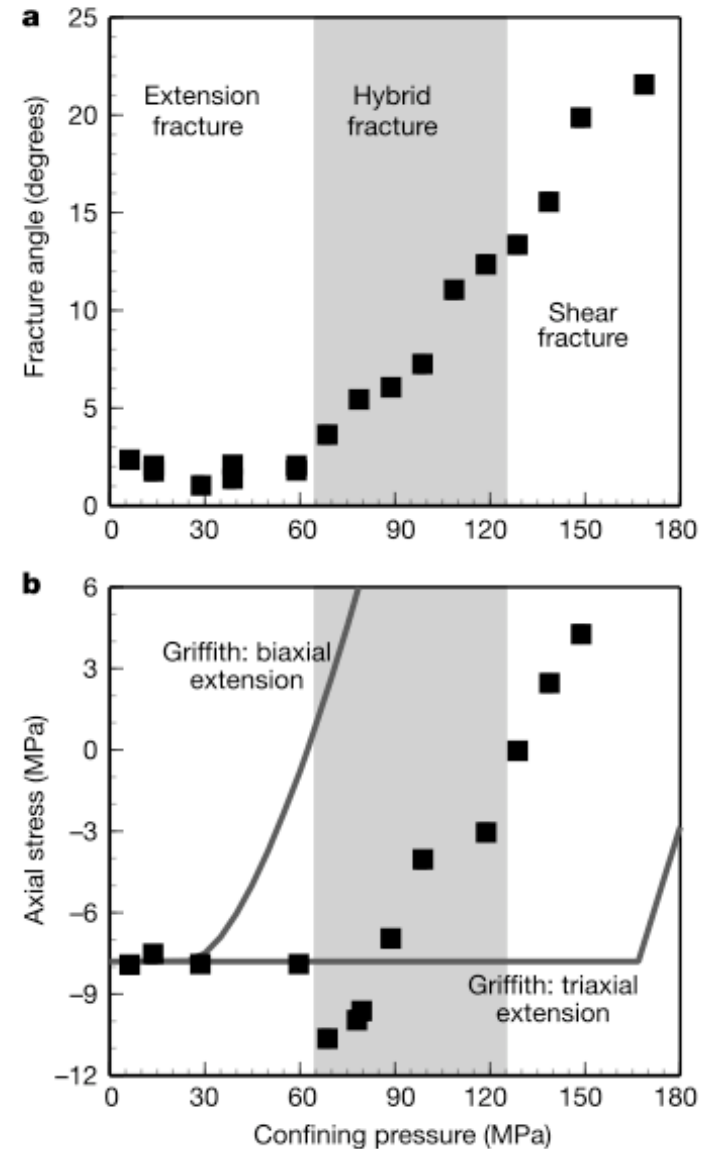
Rheology and Deformation.   Definitions.

The terms brittle and ductile can be defined in a number of ways. One def. is given by the Coulomb Failure Criterion.

Another important operational definition involves the stress-strain characteristics and the dependence of strength on mean or normal stress.
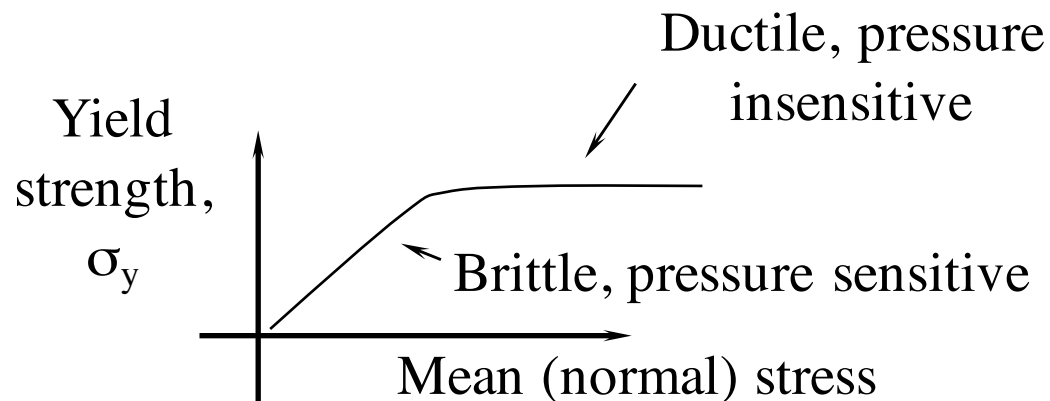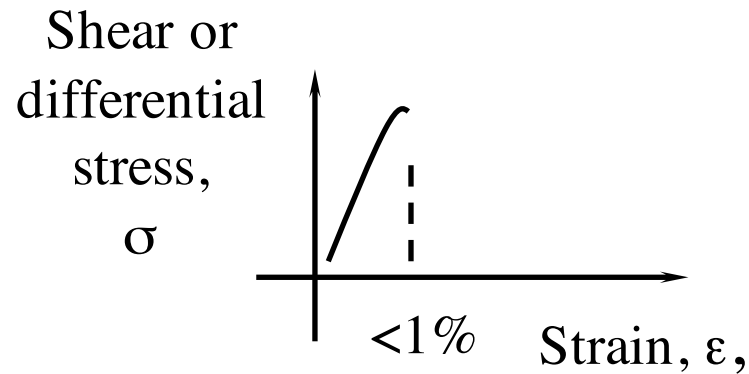

Brittle and Ductile (or plastic) deformation can be distinguished on the basis of whether the yield strength depends on pressure (mean stress or normal stress).

Why would yield strength depend on mean stress?
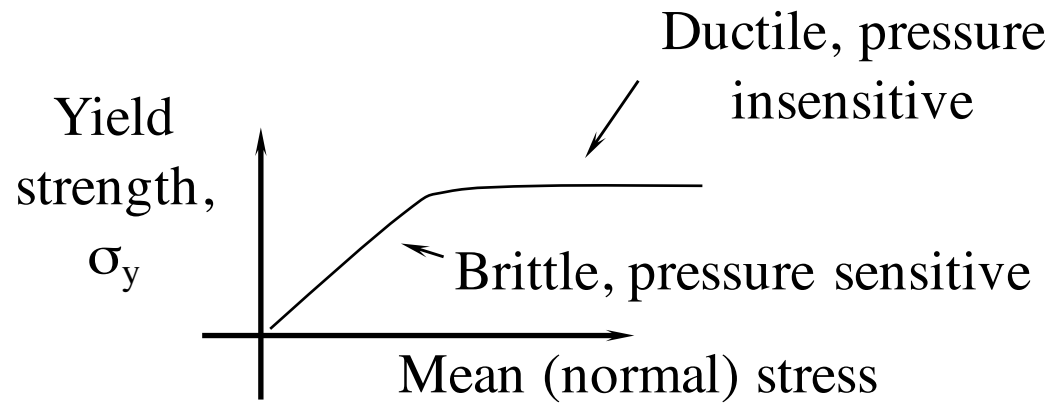
Rheology and Deformation. Definitions.

The term 'brittle' is also used to describe materials that break after very little strain.
These materials have low fracture toughness.

Shear or
differential
stress,
σ

<1%   Strain, ε,

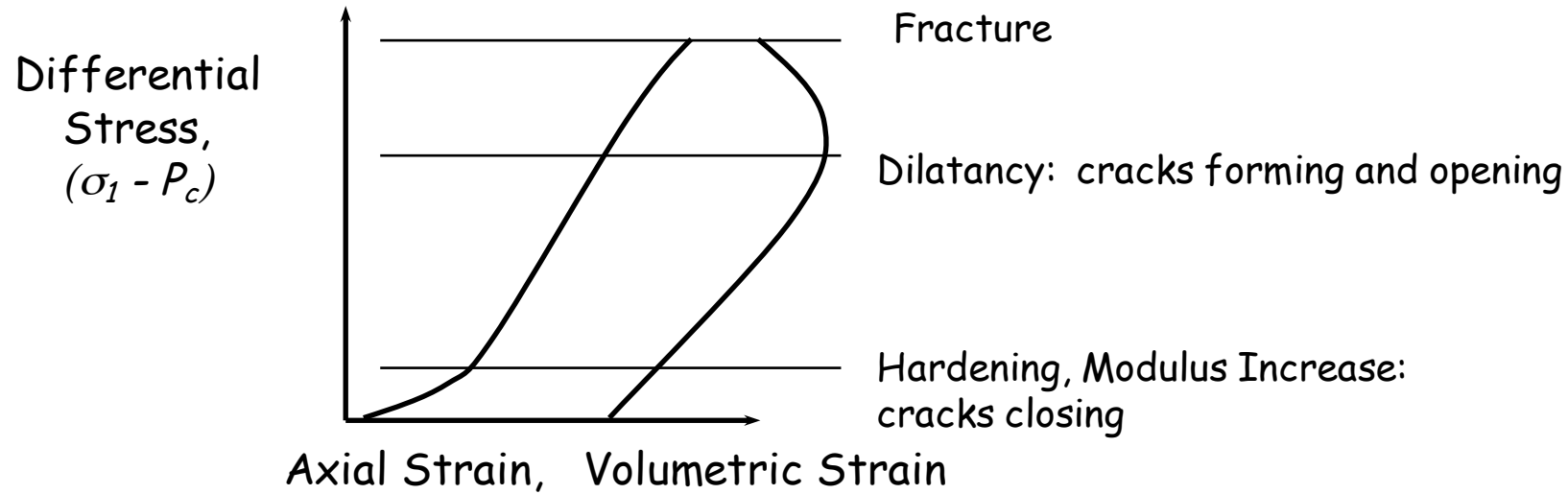Fracture toughness describes a materials ability to deform without breaking.
•Brittle materials (like glass or ceramics) have low toughness.
•Plastics have high toughness

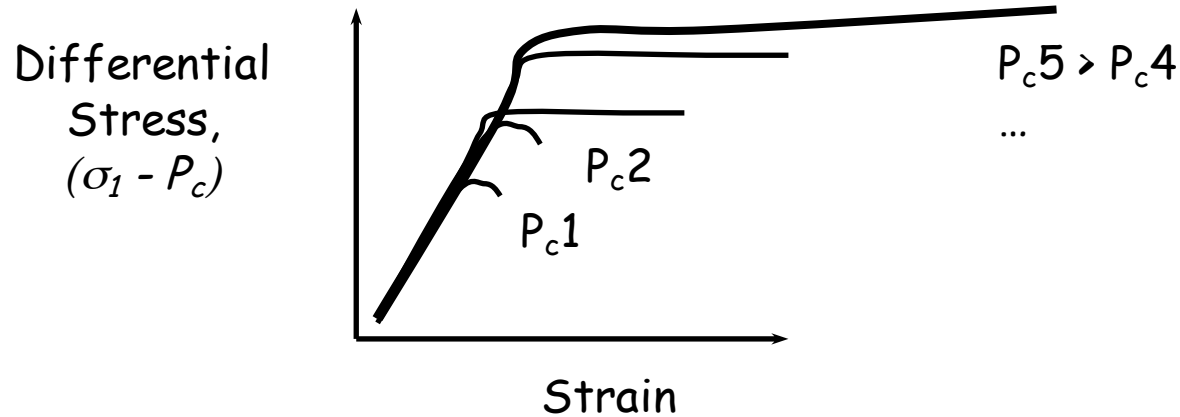**What causes the pressure sensitivity of brittle deformation?**

- Volume change. Brittle deformation involves volume change –dilatancy or compaction.
- 'Dilation' means volume increase. Dilatancy describes a shear induced volume increase. The term was introduced to describe deformation of granular materials – but dilation also occurs in solid brittle materials via the propagation of cracks.

- Work is done to increase volume against the mean stress during brittle deformation, thus the pressure sensitivity of brittle deformation.

- Ductile deformation occurs without macroscopic volume change, due to the action of dislocations. Dislocation motion allows strain accommodation.

- Brittle deformation and dilatancy

Differential Stress, $(\sigma_1 - P_c)$

Fracture

Dilatancy: cracks forming and opening

Hardening, Modulus Increase: cracks closing

Axial Strain,   Volumetric Strain

Brace, Paulding & Scholz, 1966; Scholz 1968.

Stress-strain-failure curves

Differential
Stress,
$(\sigma_1 - P_c)$

$P_c 1$

$P_c 2$

$P_c 5 > P_c 4$

...
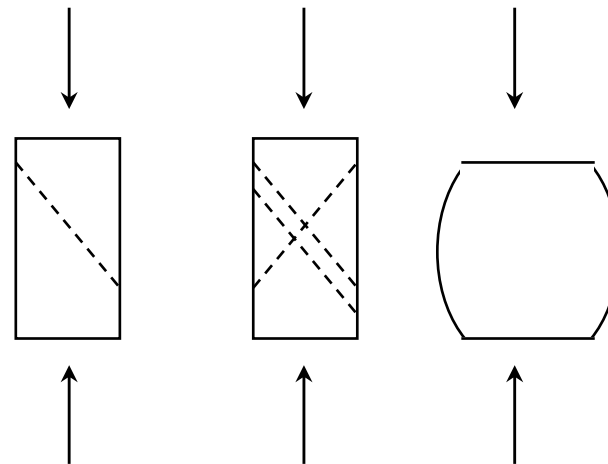
Strain

These styles can be loosely related to Brittle and Ductile deformation, respectively.
Brittle refers to pressure sensitive deformation

Brittle Failure: If we draw the stress-strain-failure curves for a range of confining pressures, we'll get a range of yield strengths, showing that σ_y is proportional to P_c.
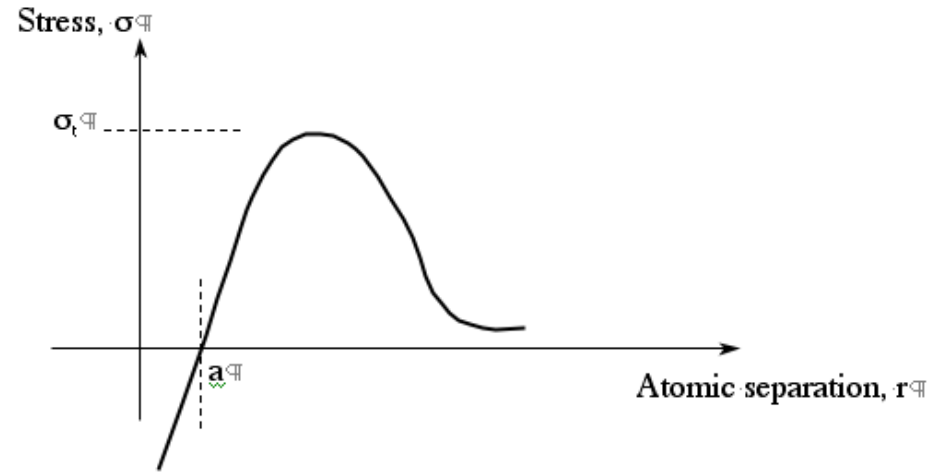
With increasing confining pressure there is a transition from localized to more broadly distributed deformation.
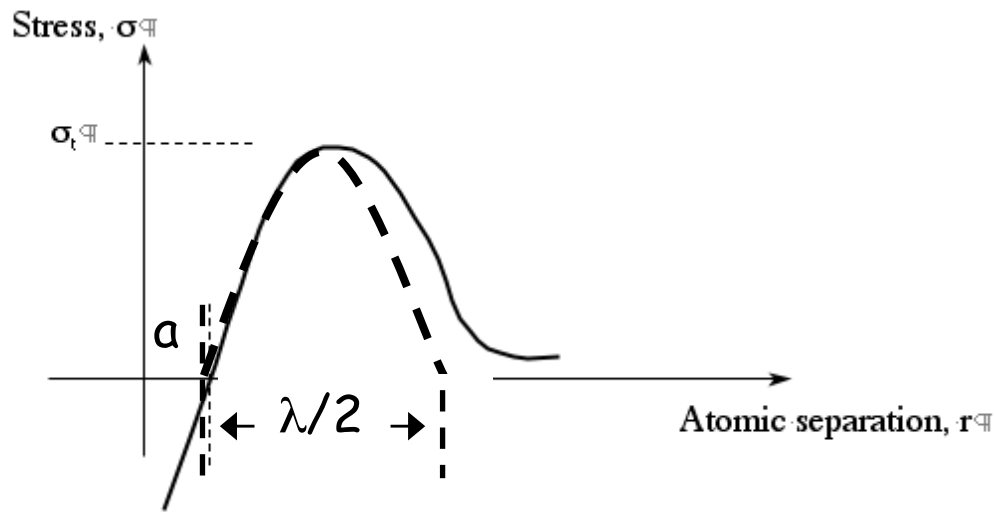
Theoretical strength of materials

- Defects
- Stress concentrations
- Griffith failure criteria
- Energy balance for crack propagation
- Stress intensity factor

Start by thinking about the theoretical strength of materials –and take crystals as a start. The strength of rocks and other polycrystalline materials will also depend on cementation strength and grain geometry so these will be more complex.
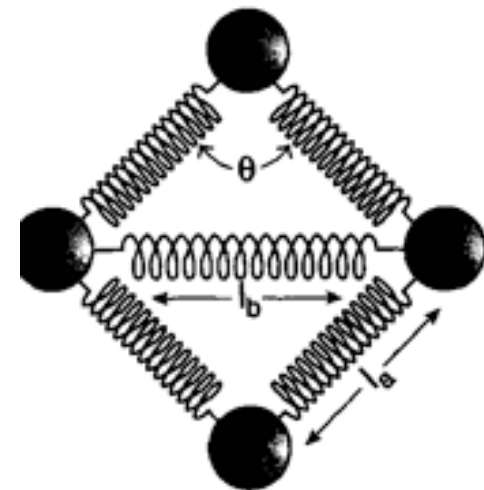
Stress, σ

σ_t

a

$\leftarrow \lambda/2 \rightarrow$

Atomic separation, r

Theoretical strength,

σ_t , of simple crystals:
Bonds must break along
a lattice plane

Consider a tensional stress field, and take *a* as the equilibrium lattice spacing.
Approximate the region around the peak strength as a sinusoid, wavelength λ
Then, for small changes in lattice spacing: the rate of stress change is related to E.

$$\sigma = \sigma_t \sin\left(\frac{2\pi(r-a)}{\lambda}\right),$$

$$\frac{d\sigma}{d(r-a)} = \frac{E}{a} = \frac{2\pi}{\lambda}\sigma_t \cos\left(\frac{2\pi(r-a)}{\lambda}\right),$$

Since $\dfrac{(r-a)}{\lambda} \ll 1$, we can re-write this as: $\sigma_t = \dfrac{E\lambda}{2\pi a}$

Tensile Strength of single x'l, by our approximation:
$$\sigma_t = \frac{E\lambda}{2\pi a}$$

The strain energy and stress is zero at thermodynamic equilibrium

Because $a \approx \lambda$, the theoretical strength is about $E/2\pi$. (See Scholz, Ch. 1.1 for additional details).

$$\sigma_t \approx \frac{E}{2\pi}$$

• This type of calculation was carried out in the early 1900's and people immediately realized that there was a problem.

• Experiments showed that E was on the order of 10's of GPa, whereas the tensile strength of most materials is closer to 10's of MPa.

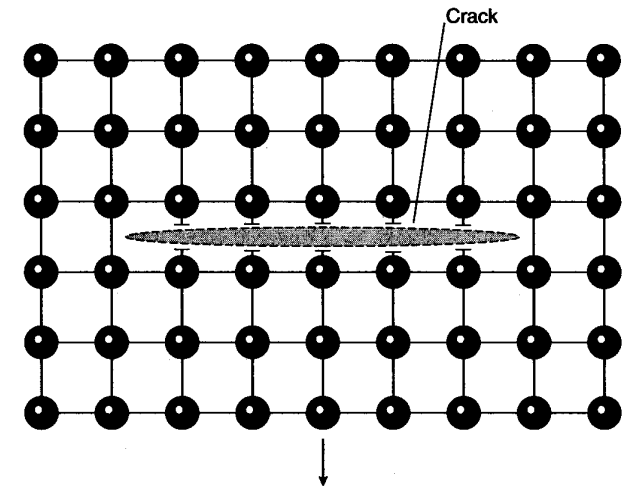Tensile Strength of single x'l, by our approximation:

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Crack

• This type of calculation was carried out in the early 1900's and people immediately realized that there was a problem.

• Experiments showed that E was on the order of 10's of GPa, whereas the tensile strength of most materials is closer to 10's of MPa.

• Griffith proposed a solution in two classic papers in the early 1920's –but the proof of his ideas had to wait until the invention of the electron microscope.

Bottom line: Defects.

Defects severely reduce the strength of brittle materials relative to the theoretical estimate. Flaws exist at all scales from atomic to the specimen size (laboratory sample size or continent scale, in the case of plate tectonics)

Stress concentrations around defects cause the local stress to reach the theoretical strength.

Two types of defects cause two types of deformation:
- cracks and crack propagation lead to brittle deformation;
- dislocations and other types of atomic misregistration lead to plastic flow and 'ductile' deformation.

Brittle deformation generally leads to catastrophic failure and separation of lattice elements.

Plastic flow produces permanent deformation without loss of lattice integrity.

Scholz generalizes these modes of deformation to make a connection with lithospheric deformation.

The upper lithosphere deforms primarily by brittle mechanisms and can be referred to as the **schizosphere** (lit. the broken part), whereas

the lower lithosphere deforms by ductile mechanisms and can be classified as the **plastosphere**.

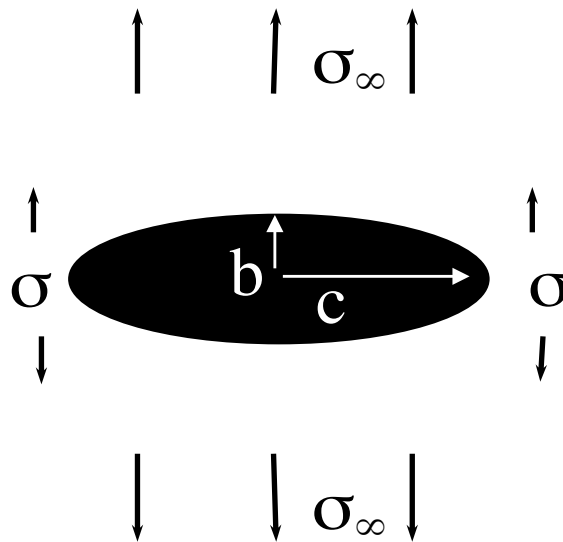**Stress concentrations around defects.**

In general, the stress field around cracks and other defects is quite complex, but there are solutions for many special cases and simple geometries

Scholz gives a partial solution for an elliptical hole in a plate subject to remote uniform tensile loading (ρ is the local curvature)

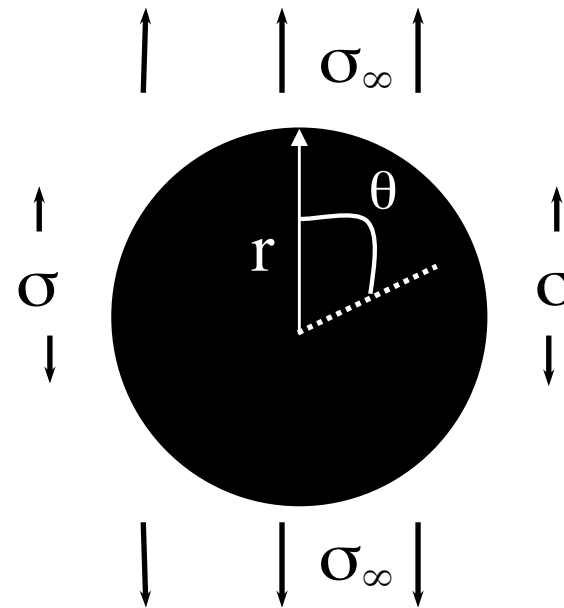$$\sigma = \sigma_\infty\left(1 + 2\frac{c}{b}\right)$$

$$\sigma = 2\sigma_\infty\left(1 + 2\sqrt{\frac{c}{\rho}}\right)$$

Crack tip stresses

Malvern (1969) gives a full solution for a circular hole of radius r = a

$$\sigma_{\theta\,Max} = \sigma_{\infty}\left(a, \frac{\pi}{2}\right) = 3\sigma_{\infty}$$

Full solution for a circular hole of radius r=a

$$\sigma_r = \frac{\sigma_{\infty}}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_{\infty}}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_{\theta} = \frac{\sigma_{\infty}}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_{\infty}}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$
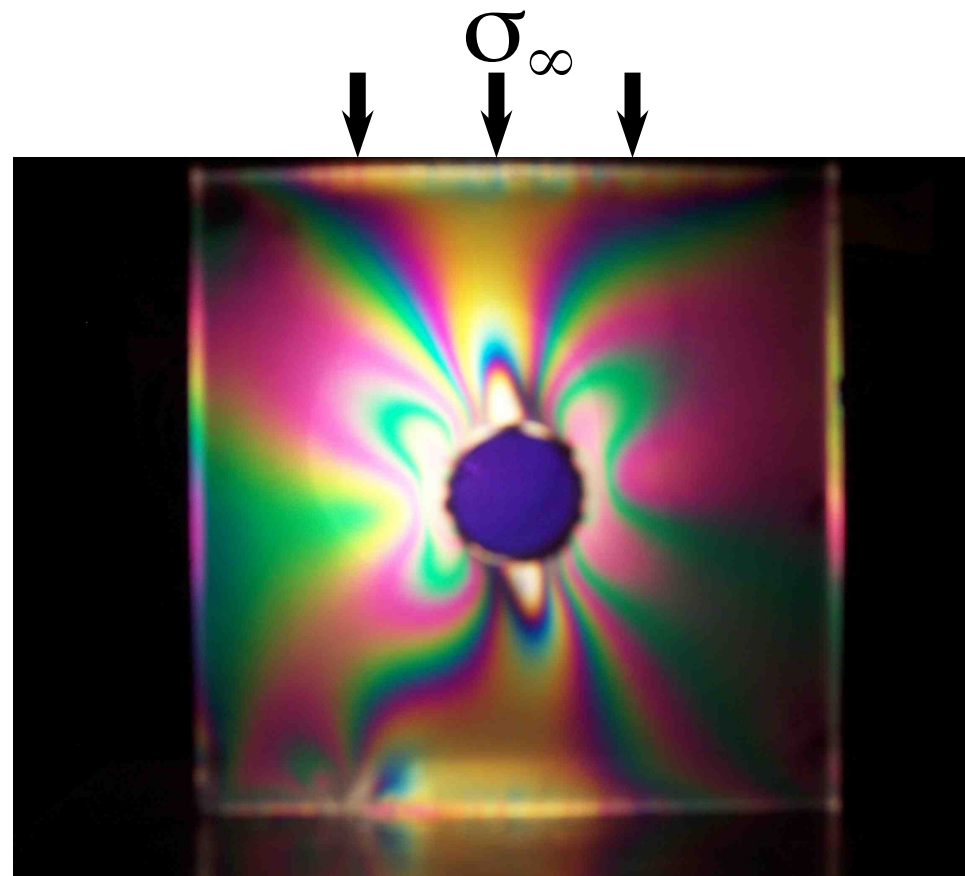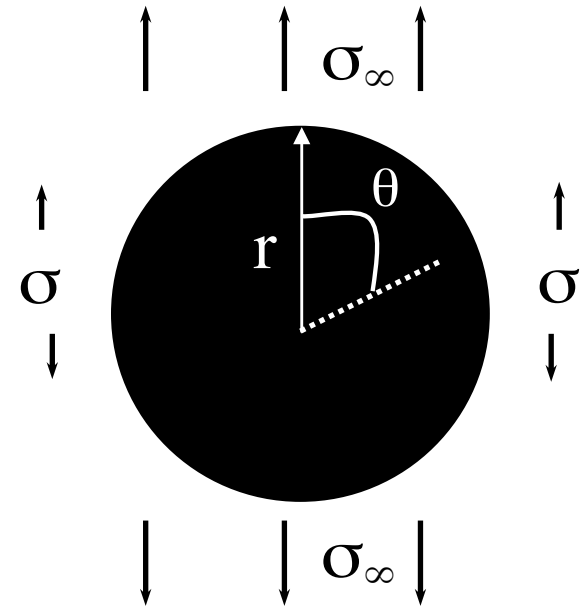
$$\tau_{r\theta} = -\frac{\sigma_{\infty}}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

Full solution for a circular hole of radius r=a

$$\sigma_r = \frac{\sigma_\infty}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_\infty}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$
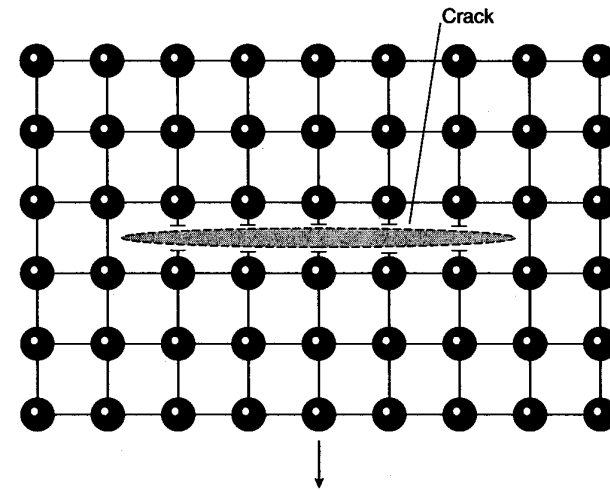
$$\sigma_\theta = \frac{\sigma_\infty}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_\infty}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

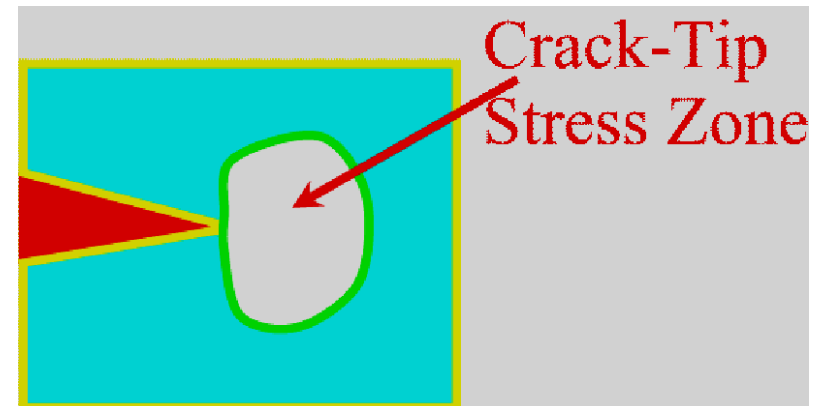$$\tau_{r\theta} = -\frac{\sigma_\infty}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

•Griffith proposed that all materials contain preexisting microcracks, and that stress will concentrate at the tips of the microcracks

•The cracks with the largest elliptical ratios will have the highest stress, and this may be locally sufficient to cause bonds to rupture

Crack

- As the bonds break, the ellipticity increases, and so does the stress concentration
- The microcrack begins to propagate, and becomes a real crack
- Today, microcracks and other flaws, such as pores or grain boundary defects, are known as Griffith defects in his honor

Crack-Tip Stress Zone

Bond separation and specific surface energy.

- Fracture involves creation of new surface area.
- The specific surface energy is the energy per unit area required to break bonds.

Two surfaces are created by separating the material by a distance $\lambda/2$ and the work per area is given by stress times displacement.

$$2\gamma = \int_0^{\frac{\lambda}{2}} \sigma_t \sin\left(\frac{2\pi(r-a)}{\lambda}\right) \, d(r-a)$$

$$= \frac{\lambda \sigma_t}{\pi}$$

This yields the estimate:
$$\gamma = \frac{Ea}{4\pi^2}$$

The surface energy is a fundamental physical quantity and we will return to it when we talk about the energy balance for crack propagation and the comparison of laboratory and seismic estimates of G, the fracture energy.

Crack mechanics and crack propagation

Fracture front

Bedding plane

"Blade-shaped" fracture

Future fracture plane
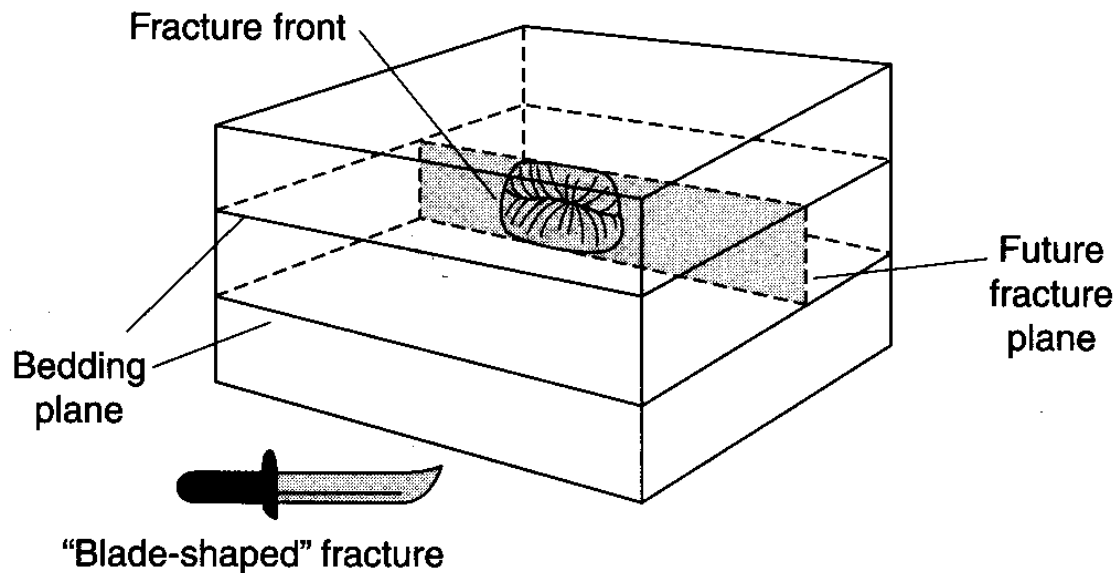
(b)

Griffith posed the problem of crack propagation at a fundamental level, on the basis of thermodynamics.

He considered the total energy of the system, including the region at the crack tip and just in front of a propagating crack.

Total energy of the system is U and the crack length is 2c, then the (cracked) solid is at equilibrium when $dU/dc = 0$

- Work to extend the crack is W
- Change in internal strain energy is U_e
- Energy to create surface area is U_s

Then: $U = (-W + U_e) + U_s$

Crack mechanics and crack propagation, Griffith theory

- Work to extend the crack is W
- Change in internal strain energy is U_e
- Energy to creation surface area is U_s

Then: $U = (-W + U_e) + U_s$

- Mechanical energy $(-W + U_e)$ decreases w/ crack extension. This is the *energy supply* during crack extension.
- $(-W + U_e)$ may come from the boundary or from local strain energy.

- The decrease in mechanical energy is balanced by an increase in surface energy (U_s is related to specific surface energy, γ, discussed above.

- The crack will extend if $dU/dc < 0$

Crack

Elastic

$\theta' < \theta$
$l_a' > l_a$
$l_b' < l_b$

(a)

(b)
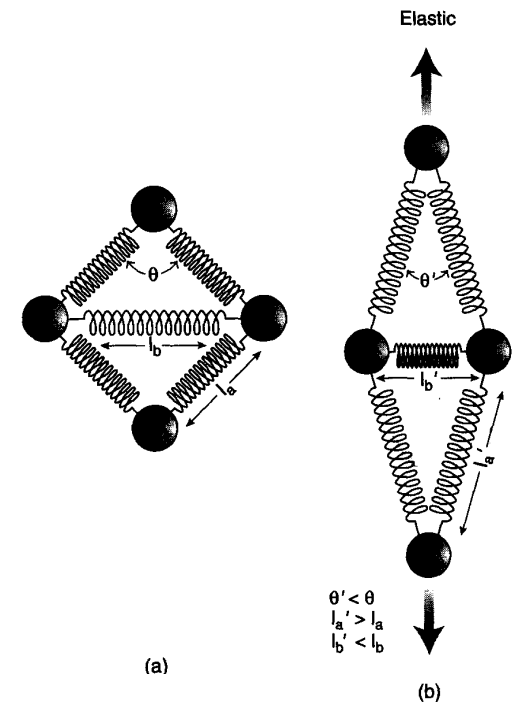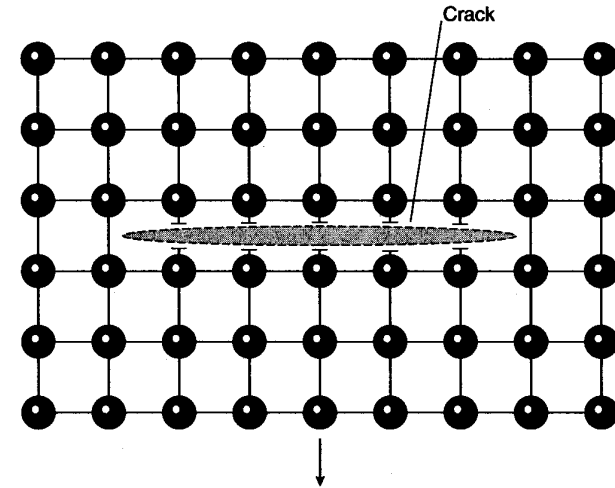
Energy balance for crack propagation, Griffith theory

$U = (-W + U_e) + U_s$

- Crack will extend if $dU/dc < 0$
- System is at equilibrium if $dU/dc = 0$

Consider a rod of length y, modulus E and unit cross section loaded in tension:

- Internal energy is: $U_e = y\sigma^2/2E$, for uniform tensile stress σ
- For a crack of length 2c, internal strain energy will increase by $\pi c^2 \sigma^2/E$

- Introduction of the crack means that the rod becomes more compliant:
    - The effective modulus is then: $E' = yE/(y+2\pi c^2)$

- The work to introduce the crack is: $W = \sigma y(\sigma/E' - \sigma/E) = 2\pi c^2 \sigma^2/E$
- Change in surface energy is $U_s = 4c\gamma$

- Thus: $U = -\pi c^2 \sigma^2/E + 4c\gamma$,
- At equilibrium: the critical stress for crack propagation (failure stress) is:
    $\sigma_f = (2E\gamma/\pi)^{1/2}$

$U = (-W + U_e) + U_s$

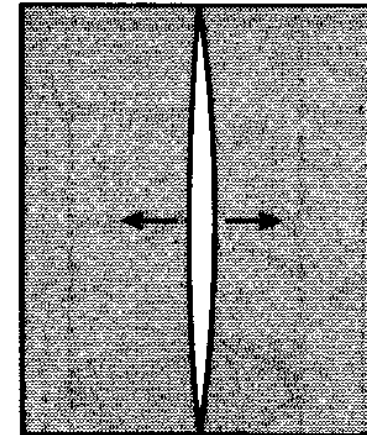- Crack will extend if dU/dc < 0
- System is at equilibrium if dU/dc = 0

- The critical stress for crack propagation (failure stress): $\sigma_f = (2E\gamma/\pi \quad)^{1/2}$
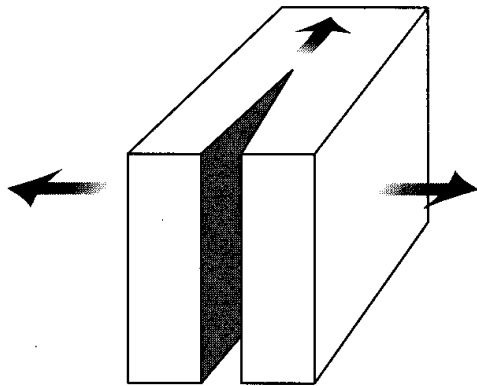
Tensile-mode cracks
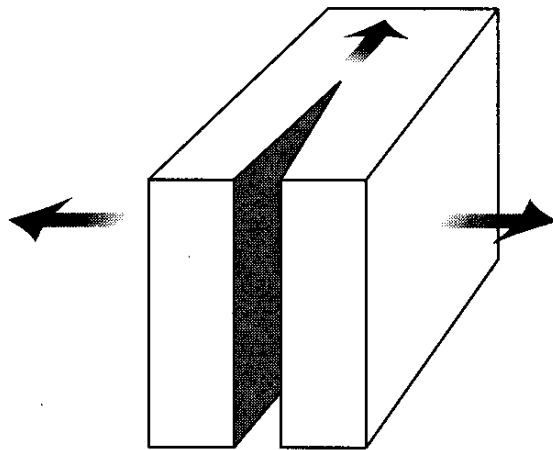
Mode I

Joint
(tensile crack)

(a)

$$\sigma_\infty = \sqrt{\frac{E\gamma}{4c}}$$

(b)

Taking σ_∞ of 10 MPa, E= 10 GPa and γ of 4×10^{-2} J/m^2, gives a crack half length c of 1 micron.

$$U = (-W + U_e) + U_s$$
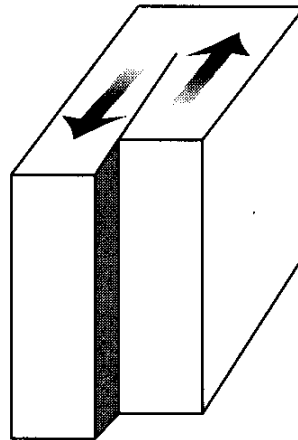
- Crack will extend if dU/dc < 0
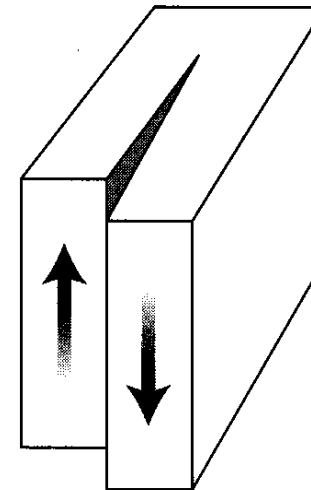- System is at equilibrium if dU/dc = 0

Tensile-mode cracks
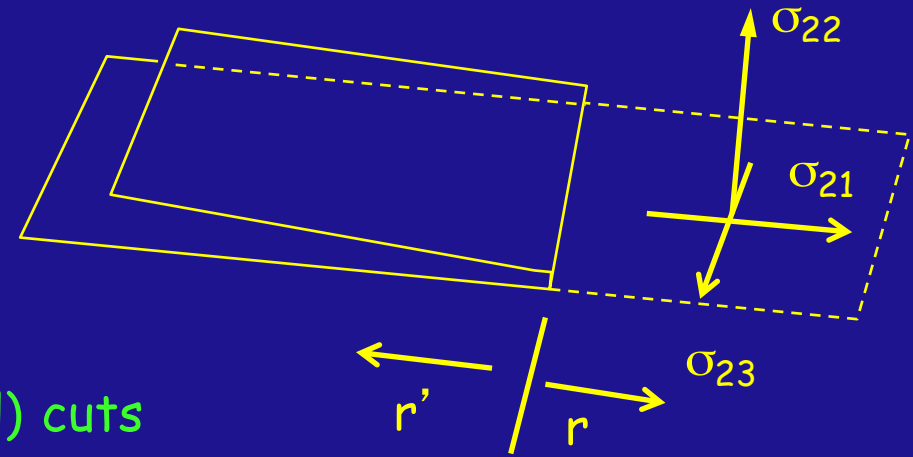Mode I

(a)

Mode II
(sliding)

(b)

Mode III
(tearing)

(c)

Fracture Mechanics and Stress intensity factors for each mode

K_I, K_{II}, K_{III}
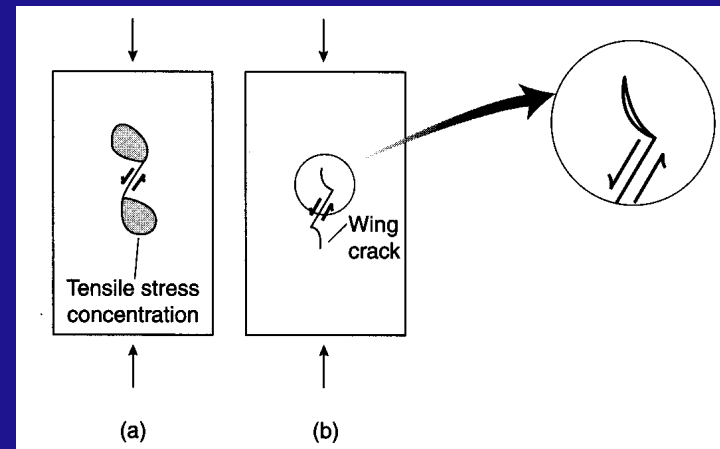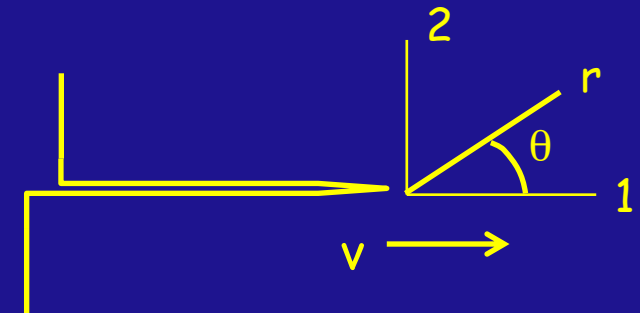
Linear Elastic Fracture Mechanics
- Frictionless cracks
- Planar, perfectly sharp (mathematical) cuts

σ_{22}

σ_{21}

σ_{23}

r'   r

Crack tip stress field written in a generalized form

$$\sigma_{ij} = K_n \frac{1}{\sqrt{2\pi r}} \; f_{ij}^n(\theta)$$

$$\begin{Bmatrix} \sigma_{22} \\ \sigma_{21} \\ \sigma_{23} \end{Bmatrix}_{tip} \approx \frac{1}{\sqrt{2\pi r}} \begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix}$$
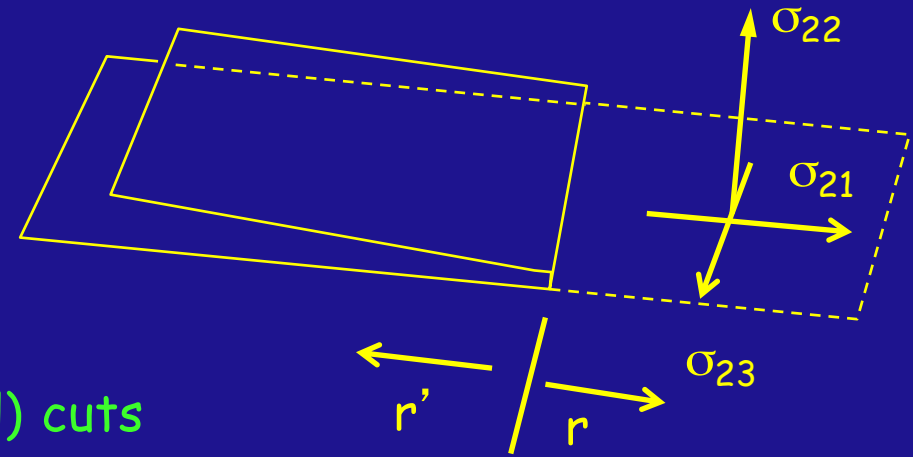
2

r

θ

1

v

Tensile stress concentration

Wing crack

(a)   (b)

Fracture Mechanics and
Stress intensity factors for each mode

K_I, K_{II}, K_{III}

Linear Elastic Fracture Mechanics
- Frictionless cracks
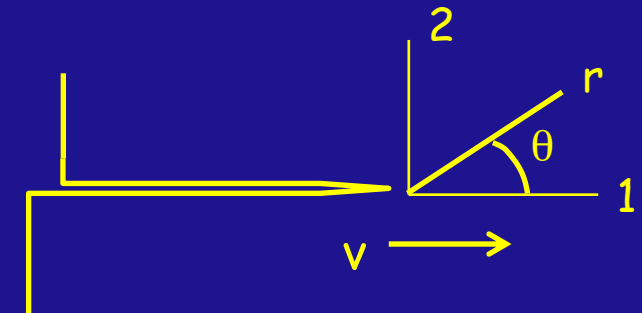- Planar, perfectly sharp (mathematical) cuts

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Crack tip stress field written in a generalized form

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2

r

θ

1

v

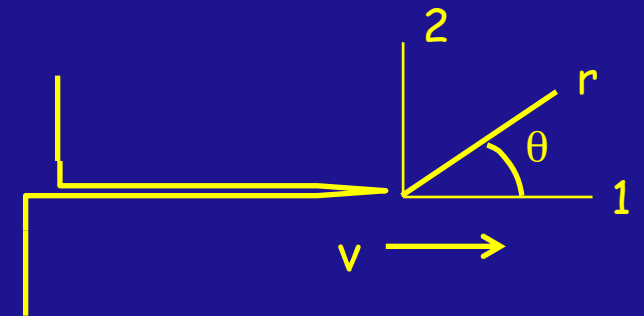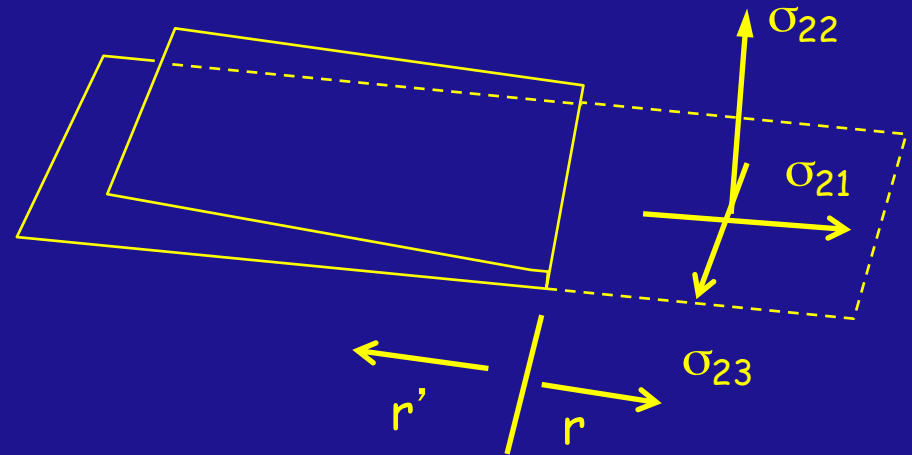Note that the functions $f_{ij}^n(\theta)$ vary from ±2, so are not major factors

For uniform remote loading of a crack of length 2c:

$$
\begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix} = \sqrt{\pi c} \begin{Bmatrix} \sigma_{22} \\ \sigma_{21} \\ \sigma_{23} \end{Bmatrix}_{applied}
$$

$$
\begin{Bmatrix} \sigma_{22} \\ \sigma_{21} \\ \sigma_{23} \end{Bmatrix}_{tip} \approx \frac{1}{\sqrt{2\pi r}} \begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix}
$$

$$
\sigma_{ij} = \frac{1}{\sqrt{2\pi r}} \; K_n
$$

$$
\sigma_{22}^{tip} = \frac{\sqrt{c}}{\sqrt{2r}} \; \sigma_{22}
$$

$$
\begin{Bmatrix} \Delta u_2 \\ \Delta u_1 \\ \Delta u_3 \end{Bmatrix} \approx \frac{4(1-\eta)}{\mu} \sqrt{\frac{r'}{2\pi}} \begin{Bmatrix} K_I \\ K_{II} \\ \dfrac{K_{III}}{(1-\eta)} \end{Bmatrix}
$$

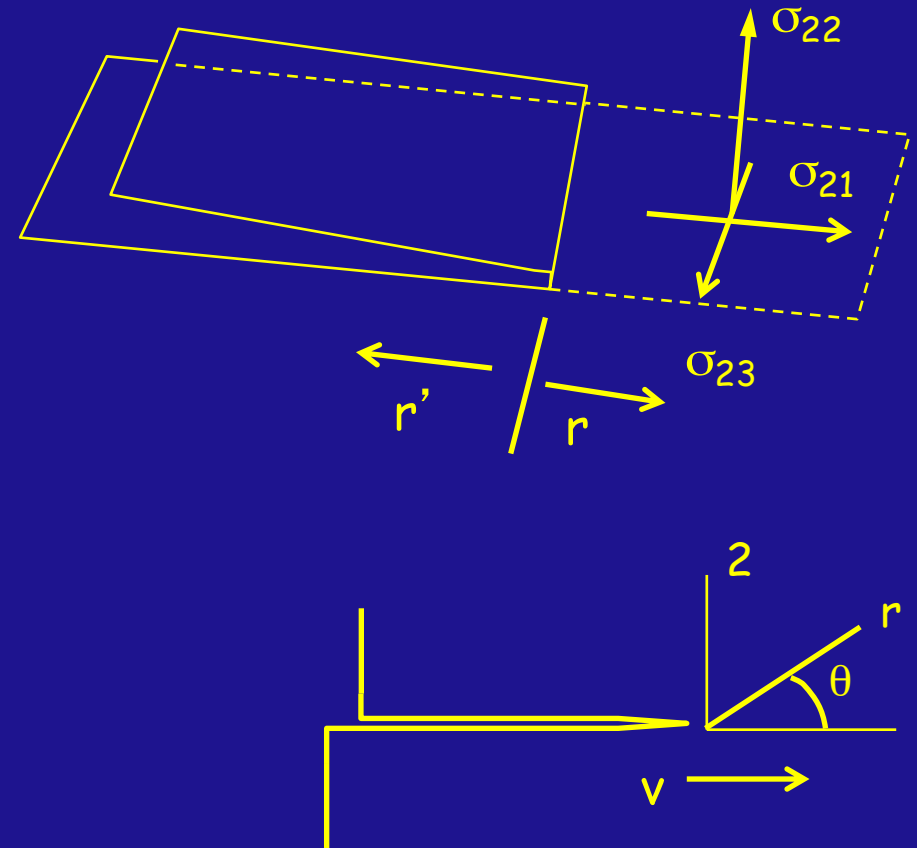# Static vs. dynamic fracture mechanics, relativistic effects

$$
\begin{Bmatrix} \Delta u_2 \\ \Delta u_1 \\ \Delta u_3 \end{Bmatrix} \approx \frac{4(1-\eta)}{\mu}\sqrt{\frac{r'}{2\pi}}\begin{Bmatrix} g_I(v)\,K_I \\ g_{II}(v)\,K_{II} \\ g_{III}(v)\,\dfrac{K_{III}}{(1-\eta)} \end{Bmatrix}
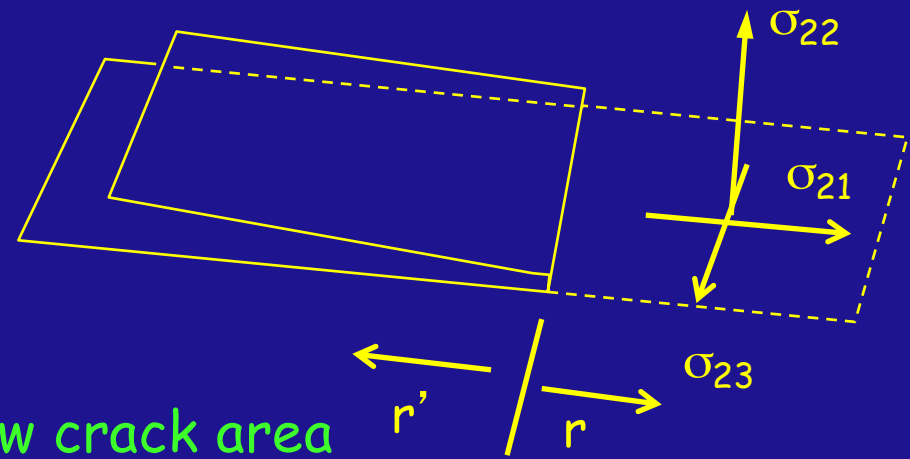$$

$$g_I(0) = g_{II}(0) = g_{III}(0) = 1 \qquad \text{Static}$$

Dynamic crack propagation

$$g_I(v) \to \infty \ \ and = g_{II}(v) \to \infty, \ as \ v \to C_R$$

$$g_{III}(v) = \frac{1}{\sqrt{1-\eta^2/C_s^2}} \to \infty, \ as \ v \to C_s$$

$$\begin{Bmatrix} \Delta u_2 \\ \Delta u_1 \\ \Delta u_3 \end{Bmatrix} \approx \frac{4(1-\eta)}{\mu} \sqrt{\frac{r'}{2\pi}} \begin{Bmatrix} g_I(v)\, K_I \\ g_{II}(v)\, K_{II} \\ g_{III}(v)\, \dfrac{K_{III}}{(1-\eta)} \end{Bmatrix}$$

σ_{22}

σ_{21}

σ_{23}

G is Energy flow to crack tip per unit new crack area

r' r

$$G \; = \; \frac{(1-\eta)}{2\mu}\left[g_I(v)K_I^2 + g_{II}(v)K_{II}^2 \right] + \frac{1}{2\mu}\, g_{III}(v)K_{III}^2$$

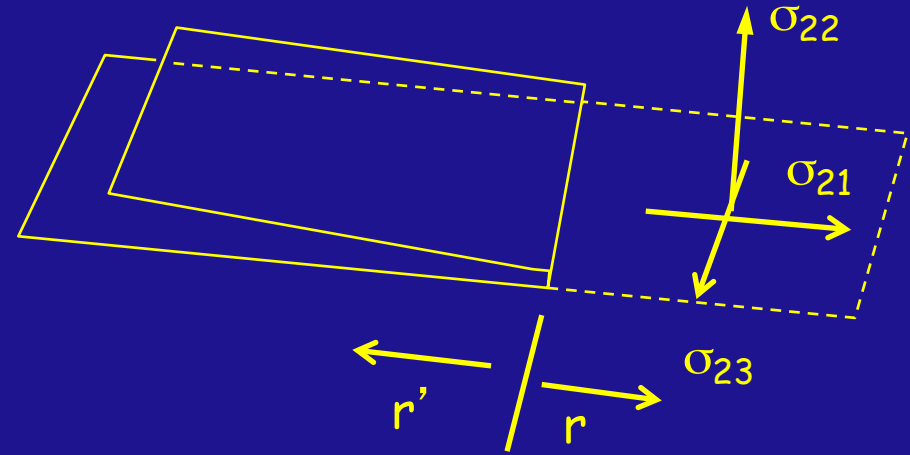$$G \; = \; G_{critical} \; = \; 2\gamma$$ Critical energy release rate

G_{crit} is a material property --the "fracture energy"

$G_{crit} = K_c^2 / E = 2\gamma$, where K_c is the critical stress intensity factor (also known as the fracture toughness).

$$\begin{Bmatrix} \sigma_{22} \\ \sigma_{21} \\ \sigma_{23} \end{Bmatrix}_{tip} \approx \frac{1}{\sqrt{2\pi r}} \begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix}$$

$$K_I = \sqrt{\pi c}\ \sigma_\infty$$

Stress field is singular at the crack tip.
- because we assumed perfectly sharp crack
- but real materials cannot support infinite stress

Process zone (Irwin) to account for non-linear zone of plastic flow and cracking
- Size of this zone will depend upon crack velocity, material properties and crack geometry
- Energy dissipation in the crack tip region helps to limit the stresses there (why?)

Crack-Tip Stress Zone